Amundi Working Paper

WP-019-2012

January 2012

Revised: July 2012

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A New Test of Mean-Variance Efficiency when All Assets Are Risky

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Abstract

The market portfolio efficiency remains controversial. This paper develops a new test of portfolio mean-variance efficiency relying on the realistic assumption that all assets are risky. The test is based on the vertical distance of a portfolio from the efficient frontier. Monte Carlo simulations show that our test outperforms the previous mean-variance efficiency tests for large samples since it produces smaller size distortions for comparable power. Our empirical application to the U.S. equity market highlights that the market portfolio is not mean-variance efficient, and so invalidates the zero-beta CAPM.

Keywords: Efficient portfolio, mean-variance efficiency, efficiency test.

JEL codes: G11, G12, C12.

1. Introduction

This paper proposes a new test of portfolio mean-variance (MV) efficiency based on the realistic assumption that all assets are risky. Testing the mean-variance (MV) efficiency of the market portfolio, or equivalently testing the validity of the Capital Asset Pricing Model (CAPM) of Sharpe (1964) and Lintner (1965), is a major task for financial econometricians. The debate on this issue dates back to the breakthrough theoretical contributions of Roll (1977) and Ross (1977) questioning the efficiency of the market portfolio. In the wake of these contributions, numerous empirical studies (Gibbons, 1982; Gibbons *et al.*, 1989; MacKinlay and Richardson, 1991; among others) found that the market portfolio may indeed lie far away from the efficient frontier. Ironically, this debate was recently fuelled by Levy and Roll (2010), who published an article entitled "*The market portfolio may be mean-variance efficient after all*". We take a fresh look at this issue.

All portfolio managers are—or should be—faced with the issue of checking whether a given portfolio is optimal within a predefined investment universe. For this purpose, MV efficiency, as defined by Markowitz (1952, 1959), remains the key optimality concept. Currently, the econometric literature offers a wide variety of tests for MV efficiency. Most are designed for universes that include a riskless asset.[1] This represents a considerable constraint when it comes to practical implementation. By contrast, this paper focuses on MV efficiency tests that allow all assets to be risky.

The assumption that all assets are risky is highly relevant given that riskless assets are no longer realistic in modern financial markets. The recent debt crisis has highlighted that even the supposedly safest assets, namely sovereign bonds issued by developed countries, are

[1] When the investment universe includes a riskless asset, the efficient frontier is a straight line, which makes the derivations far simpler (Gourieroux *et al.*, 1997). Tests falling in this category have been proposed by Gibbons (1982), Jobson and Korkie (1982), and MacKinlay and Richardson (1991), among others. The test introduced by Gibbons *et al*. (1989) has since then become the standard. Michaud (1989) and Green and Hollifield (1992) discuss the limitations of this framework. Besides, MV efficiency tests must be distinguished from MV spanning tests, which examine whether the efficient frontier built from a given set of assets intersects the frontier resulting from a larger set (see De Roon and Nijman (2001) for a survey).

exposed to default risk. In the same way, the freezing of the money markets and the Lehman Brothers' bankruptcy underlined the counterparty and liquidity risks associated with money market investments (Bruche and Suarez, 2010; Krishnamurthy, 2010; Acharya *et al.*, 2011). Investors can thus meet severe restrictions on borrowing (Black, 1972), and the riskless borrowing rate can largely exceed the Treasury bill rate (Brennan, 1971). For all these reasons, MV efficiency is better tested without assuming the availability of a riskless asset.

Two broad classes of MV efficiency tests for risky-asset universes exist in the literature: likelihood-based tests and geometric tests. The likelihood-based tests are directly inspired by the formulation of the CAPM. While the riskless asset is needed to establish the original CAPM, further refinements by Black (1972) allow the riskless asset to be replaced by the zero-beta portfolio. To address the nonlinearities embedded in the Black CAPM, Gibbons (1982) builds a likelihood-ratio test statistic, for which Kandel (1984, 1986) derives the exact asymptotic chi-square distribution. However, because this test uses the Gauss-Newton algorithm, practical implementation turns out to be complex (Zhou, 1991). Moreover, Shanken (1985) shows that Gibbons' (1982) test tends to over-reject MV efficiency in finite samples.[2] Levy and Roll (2010) (henceforth, L&R) offer a novel likelihood-ratio test for MV efficiency. This test is based on implicitly estimating the zero-beta rate by determining the minimal changes to sample parameters that make a market proxy efficient.[3]

On the other hand, the first geometric test of Basak, Jagannathan and Sun (2002) (henceforth, BJS) is based on the "horizontal distance" between the portfolio whose MV efficiency is in question and its same-return counterpart on the MV efficient frontier.[4] Unfortunately, some portfolios lack such a counterpart (Gerard *et al.*, 2007), which in turn limits the applicability

[2] In reaction to these criticisms, several authors (Shanken, 1985, 1986; Zhou, 1991; Velu and Zhou, 1999; Beaulieu *et al.*, 2008) provide lower and upper bounds to the test p-values.
[3] Small variations in expected returns and volatilities may indeed lead to significant changes in the MV efficient frontier (Best and Grauer, 1991; Britten-Jones, 1999).
[4] The null hypothesis is that the "horizontal distance" is zero. BJS derive the asymptotic distribution of this distance. Interestingly, the BJS test can be implemented with and without restrictions on short-selling. Besides, the BJS test can also be used to compare efficient frontiers (Ehling and Ramos, 2006; Drut, 2010).

of the BJS test. Moreover, while this horizontal test is particularly suitable in the case of investors seeking to minimize the risk of their investments with an expected return goal, this is not the case for all categories of investors. Some of them have instead a well-defined objective of risk they cannot afford to go beyond. They will thus try, given that risk constraint, to obtain the highest possible return. This is the case, for example, of benchmarked portfolio managers, which represent a substantial part of the asset management industry. Their objective is to maximize the excess return of the portfolio over the benchmark and at the same time make sure that the risks do not exceed a given “tracking error” fixed in the objectives of the funds (Roll, 1992; Jorion, 2003). The vertical test proposed in this paper allows to address an audience of investors with different objectives and circumvents the aforementioned limitations. It is based on the vertical inefficiency measure proposed by Kandel and Stambaugh (1995), Wang (1998), and Li *et al.* (2003), namely the difference between the portfolio’s expected return and the expected return of its same-variance counterpart on the MV efficient frontier. Both tests are in fact complementary. As for testing the efficiency of the market portfolio, where both dimensions (return and risk) should be simultaneously taken into account, the vertical and horizontal tests could be used simultaneously.

Our contribution is twofold. First, we define the vertical test statistic for MV efficiency, establish its asymptotic distribution, and compare its size and power performances to those of the L&R and BJS tests through Monte Carlo simulations. While no clear hierarchy emerges for small samples, the vertical test outperforms its competitors for large samples as it exhibits equivalent power for a smaller size. Secondly, we re-examine the market portfolio MV efficiency using the three tests under review (L&R, BJS and the vertical tests). Irrespectively of the number of stocks in the universe, we find that the market portfolio is never MV efficient according to both the BJS and the vertical tests. For the L&R test, the conclusion depends on the value given to the coefficient α, which determines the relative weight assigned to sample mean changes against standard deviation changes. In other words, the L&R test reaches no clear-cut and definitive conclusion regarding the market portfolio efficiency. Although still frail, the evidence points to the inefficiency of the market portfolio, supporting the Roll's (1977) critique of the CAPM.

The paper is organized as follows. Section 2 presents the vertical test and its asymptotic properties. Section 3 assesses the size and power of the vertical test and its two competitors. Section 4 tests the Black CAPM on the U.S. equity market. Section 5 concludes.

2. The Vertical Test of Mean-Variance Efficiency

Consider an investment universe composed of N primitive assets with stationary returns characterized by a N-dimensional vector R, with $E(R)=\mu$, and $Cov(R)=\Sigma$. The tested portfolio, P, is composed of primitive assets. Let r denote its return, with $E(r)=\beta$ and $Var(r)=\nu^2$.

Given a sample of returns of size T denoted $(R_t)_{t=1..T}$ for the N primitive assets and $(r_t)_{t=1..T}$ for portfolio P, the empirical counterparts of parameters μ, Σ, β, and ν^2 are respectively given by:

$$\hat{\mu}=\frac{1}{T}\sum_{t=1}^{T}R_t \tag{1}$$

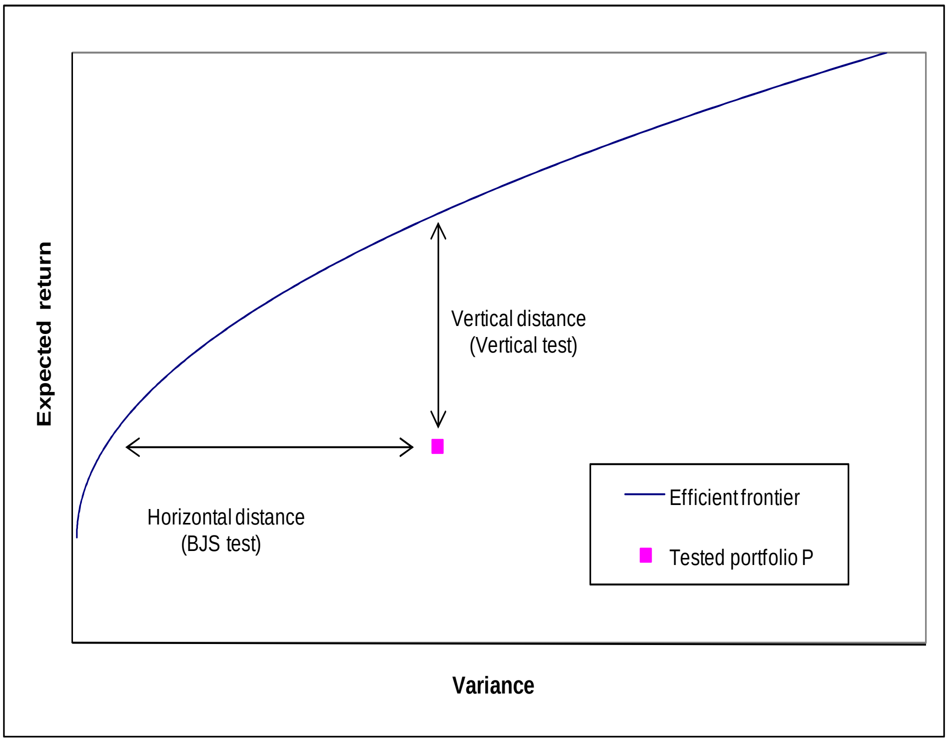
$$\hat{\Sigma}=\frac{1}{T}\sum_{t=1}^{T}R_t R_t'-\hat{\mu}\hat{\mu}' \tag{2}$$

$$\hat{\beta}=\frac{1}{T}\sum_{t=1}^{T}r_t \tag{3}$$

$$\hat{\nu}^2=\frac{1}{T}\sum_{t=1}^{T}(r_t-\hat{\beta})^2 \tag{4}$$

where R_t and r_t are the date-t returns on the N primitive assets and on portfolio P, respectively.

Figure 1. Horizontal and vertical distances between portfolio *P* and the efficient frontier



As illustrated by Figure 1, the horizontal distance underlying the BJS test measures of portfolio P inefficiency is the difference between the variance of P and the variance of its same-expected-return counterpart on the efficient frontier.

Our vertical test is conceived by transposing the BJS (2002) methodology to the vertical inefficiency measure introduced by Kandel and Stambaugh (1995), Wang (1998), and Li *et al.* (2003). Hence, the vertical test statistic[5] is the distance between the expected return of portfolio P and the expected return of its same-variance MV efficient counterpart. The estimated distance, denoted by $\hat{\theta}$, is the solution to the following program:

[5] Another possibility would be to take the minimal Euclidian distance between portfolio P and the efficient frontier. This approach would certainly be more elegant, but would also be much more tedious as it would mix up first and second order parameters.

$$\hat{\theta} = \begin{bmatrix} \min_{\omega} \left\{ \hat{\beta} - \omega' \hat{\mu} \right\} \\ s.t.\ \omega' \hat{\Sigma} \omega = \hat{v}^2 \\ \sum_{i=1}^{p} \omega_i = 1,\ \omega_i \geq 0,\ for\ i = 1,...,p \end{bmatrix} \tag{5}$$

The following proposition states that, under the null that portfolio P is MV efficient, estimator $\hat{\theta}$ asymptotically follows a normal distribution.

Proposition 1

$\hat{\theta}$ asymptotically follows a normal distribution:

$$\sqrt{T}(\hat{\theta} - \theta) \to N(0, \phi^2) \text{ as } T \to \infty. \tag{6}$$

with $\phi^2 = \lim_{T \to \infty} \frac{\partial \hat{\theta}}{\partial \overline{V}} \Delta \frac{\partial \hat{\theta}}{\partial \overline{V}}$, where Δ is given by Equation (A2) in Appendix A and represents the asymptotic covariance matrix of the distinct elements of $\hat{\mu}$, $\hat{\Sigma}$, $\hat{\beta}$, and $\hat{v}$, and $\left(\frac{\partial \hat{\theta}}{\partial \overline{V}} \right)$ is given by (A6) in Appendix A.

Proof: See Appendix A.

As for the BJS test, this asymptotic result does not require normality assumptions on the asset returns.[6] Moreover, as demonstrated in Appendix A, this result holds both with and without short-selling restrictions.

3. Power and Size Performances

In this section, we assess the size and power of the vertical test and compare its performances to those of the BJS and L&R tests. To this end, we simulate series of returns drawn from the

[6] Here, returns are assumed identically and independently distributed. The impact of autocorrelation and heteroskedasticity in returns could be investigated thanks to the block bootstrap methodology, along the lines of Topaloglou and Scaillet (2010).

investment universe imagined by Das *et al.* (2010), including three assets with jointly normal returns having the following parameters:

$$\mu = \begin{bmatrix} 0.05 \\ 0.10 \\ 0.25 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 0.0025 & 0.0000 & 0.0000 \\ 0.0000 & 0.0400 & 0.0200 \\ 0.0000 & 0.0200 & 0.2500 \end{bmatrix} \tag{7}$$

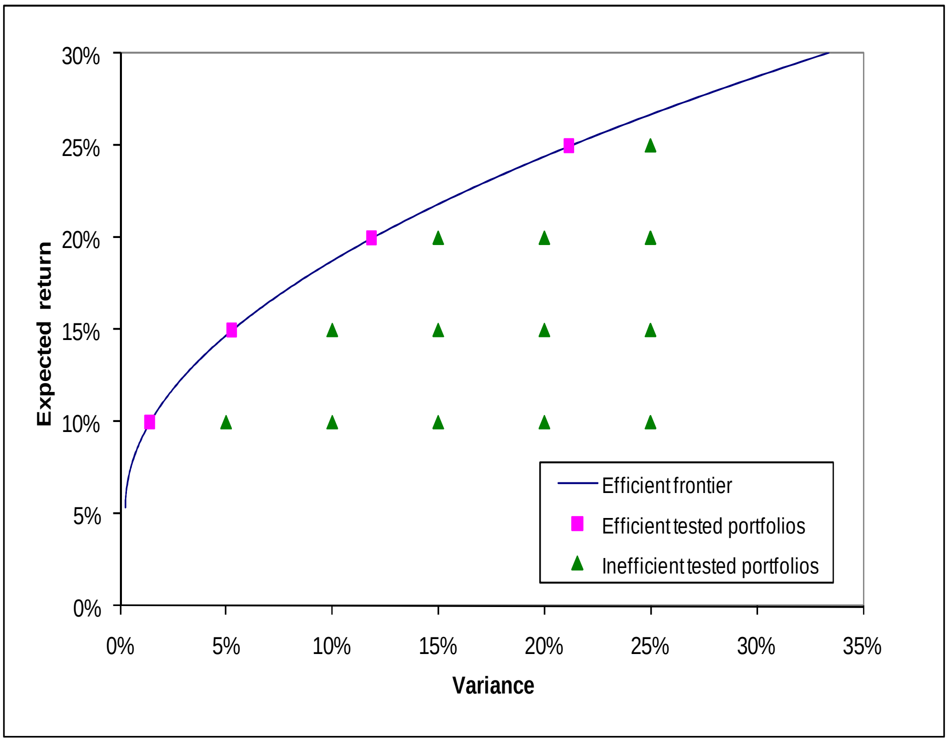
Das *et al.* (2010) interpret the first asset as a bond, the second as a low-risk stock, and the third as a highly speculative stock. For the sake of comparability,[7] we focus here on the case where short-selling is allowed.

We simulated 1,000 series of returns of lengths 60, 120, 180, and 240, respectively. In each case, two groups of portfolios were composed. The portfolios in the first group were generated on the efficient frontier in order to estimate the risk of type I error (false rejection of the true hypothesis that portfolios are mean-variance efficient). The portfolios in the second group were generated below the efficient frontier to estimate the risk of type II error (failure to reject the false hypothesis).

We follow the assessment of statistical tests suggested by Wasserman (2004). This procedure is based on power maximization (i.e., minimization of the risk of type II error) for a given small size (i.e., risk of type I error). Figure 2 features all tested portfolios on a grid in the MV plane. To each of them, we successively apply the BJS, L&R, and vertical tests.

[7] L&R solely apply their test to cases where short-selling is allowed. Actually, the performances of their test when short-selling is restricted have not been investigated so far.

Figure 2. Efficient frontier and tested portfolios



BJS measure the difference in variances λ between the tested portfolio P and its MV efficient counterpart with same expected return. The estimated horizontal distance is the solution to the following program:

$$\hat{\lambda} = \left[\begin{array}{l} \min\limits_{\omega} \left\{ \omega' \hat{\Sigma} \omega - \hat{v}^2 \right\} \\ s.t.\ \omega' \hat{\mu} = \hat{\beta} \\ \sum\limits_{i=1}^{p} \omega_i = 1,\ \omega_i \geq 0,\ for\ i = 1, ..., p \end{array} \right] \tag{8}$$

Under the null that portfolio P is MV efficient, $\hat{\lambda}$ asymptotically follows a normal distribution: $\sqrt{T}\left(\hat{\lambda} - \lambda\right) \rightarrow N(0, \varepsilon^2)$ as $T \rightarrow \infty$.

The L&R test draws on the evidence that slight variations in the sample parameters may make a portfolio MV efficient. More precisely, the L&R test statistic is built from asset-return parameters (μ^*, σ^*) that minimize a given distance to the sample parameters $(\hat{\mu}, \hat{\sigma})$ while making portfolio *P* MV efficient:

$$(\mu^*, \sigma^*) = \underset{(\mu,\sigma)\in\left(\Re^N \times \Re^{+N}\right)}{\arg\min} \quad d\left((\mu,\sigma),(\hat{\mu},\hat{\sigma})\right) \tag{9}$$

where distance *d* is defined by:

$$d\left((\mu,\sigma),(\hat{\mu},\hat{\sigma})\right) = \sqrt{\alpha \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\mu_i - \hat{\mu}_i}{\hat{\sigma}_i}\right)^2 + (1-\alpha)\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\sigma_i - \hat{\sigma}_i}{\hat{\sigma}_i}\right)^2} \tag{10}$$

and α is a coefficient determining the relative weight assigned to deviations in means relative to the deviations in standard deviations.[8]

For simplicity, L&R reduce the number of parameters to estimate by imposing that covariance matrix Σ^* computed from (μ^*, σ^*) is based on the sample correlation matrix:

$$\Sigma^* = \begin{bmatrix} \sigma^*_1 & 0 & \cdots & & 0 \\ \vdots & \sigma^*_2 & & & \\ & & \ddots & & \vdots \\ & & & & 0 \\ 0 & & \cdots & 0 & \sigma^*_N \end{bmatrix} \hat{C} \begin{bmatrix} \sigma^*_1 & 0 & \cdots & & 0 \\ \vdots & \sigma^*_2 & & & \\ & & \ddots & & \vdots \\ & & & & 0 \\ 0 & & \cdots & 0 & \sigma^*_N \end{bmatrix} \tag{11}$$

where $\hat{C}$ is the sample correlation matrix. In that way, only the variances have to be estimated.

Under the hypothesis that the *N* original assets follow a jointly normal distribution, the likelihood ratio is given by:

$$T\left\{\log\left(\frac{\hat{\Sigma}}{\Sigma^*}\right) - N + trace\left(\hat{\Sigma}^{-1}\left(\Sigma^* + (\mu^* - \hat{\mu})(\mu^* - \hat{\mu})'\right)\right)\right\} \tag{12}$$

[8] See Equation (2) in L&R (2010).

This test statistic asymptotically follows a chi-square distribution with $2N$ degrees of freedom.[9]

The choice of the trade-off parameter α in Equation (10) is instrumental to the implementation of the L&R test. Indeed, a low (resp. high) value of α would create a bias towards standard deviations (resp. means). In extreme cases ($\alpha = 0$ and $\alpha = 1$), the asymptotic distribution of the L&R test statistic degenerates into a chi-square with N degrees of freedom. In our performance assessments, we follow L&R and set the value of α to 0.75.

3.1. False Rejection of Efficient Portfolios

We first assess the type I error. The four simulated efficient portfolios have expected returns of 10%, 15%, 20% and 25%, respectively. The rejection frequencies of the null of portfolio efficiency at the 5% probability level are displayed in Table 1.[10] The results show that the size is uniformly the lowest for the vertical test, followed by the L&R test. Nevertheless, the vertical test, and to a lesser extent the L&R test, exhibit rejection frequencies that lie below the theoretical threshold of 5%.

[9] It should be noticed that L&R do not take into account that μ^* and σ^* are sample-dependent (as is the case since the determination of μ^* and σ^* results from the minimization of distance $D\left((\mu,\sigma),(\mu,\sigma)^{sam}\right)$, which depends on the sample). As a consequence, the fact that the asymptotic distribution of the test statistic should follow a chi-squared distribution may be questioned.

[10] The results for the 1% and 10% probability levels are given in Table B1 in Appendix B.

Table 1. Rejection frequencies (in percent) at the 5% probability level for the efficient portfolios

Expected return		T	BJS	Vertical	L&R
	10%	60	7.6	0.6	3.7
		120	5.5	0.4	1.8
		180	5.1	0.4	1.4
		240	4.1	0.2	1.3
	15%	60	6.1	0.6	2.9
		120	6.4	0.4	1.9
		180	5.1	0.0	1.3
		240	4.6	0.0	1.5
	20%	60	8.6	0.6	3.1
		120	5.8	0.4	1.7
		180	5.4	0.3	1.5
		240	4.6	0.2	1.6
	25%	60	6.4	0.6	2.8
		120	6.3	0.4	1.7
		180	5.6	0.0	1.5
		240	4.9	0.0	0.0

Note: BJS: Basak *et al*. (2002) test; Vertical: vertical test; L&R: Levy and Roll (2010) test. T is the sample size.

3.2. Rejection of Inefficient Portfolios

We now apply the three MV efficiency tests under review to thirteen portfolios simulated as inefficient in order to assess the probability of falsely concluding that the portfolio was efficient. The results are given in Table 2 for 5% probability.[11]

Table 2. Rejection frequencies (in percent) at the 5% probability level for the inefficient portfolios

		Variance														
		5%			10%			15%			20%			25%		
Expected return	T	BJS	Vertical	L&R	BJS	Vertical	L&R	BJS	Vertical	L&R	BJS	Vertical	L&R	BJS	Vertical	L&R
10%	60	89.8	49.1	66.6	94.4	62.0	76.4	96.8	69.0	76.7	96.8	70.3	79.9	98.2	72.3	80.6
	120	99.2	85.4	93.9	100.0	93.4	96.4	100.0	94.7	96.2	100.0	96.6	95.9	99.7	96.4	96.1
	180	100.0	96.7	99.1	100.0	98.9	99.6	100.0	99.5	99.7	100.0	99.3	99.3	100.0	99.9	99.4
	240	100.0	99.5	99.9	100.0	99.7	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.9
15%	60				71.5	24.8	35.1	86.5	38.0	55.4	89.4	49.9	66.7	93.8	55.4	72.3
	120				92.1	51.7	64.5	98.5	72.6	87.2	99.1	83.1	92.5	99.7	86.8	94.9
	180				98.8	75.3	86.5	99.6	92.7	97.4	100.0	96.2	98.9	100.0	97.6	99.5
	240				99.8	88.9	93.8	100.0	97.9	99.5	100.0	99.2	99.9	100.0	99.7	99.9
20%	60							35.6	5.2	5.9	64.5	19.2	27.2	75.7	28.5	44.6
	120							56.3	12.9	12.2	84.2	41.7	53.0	93.8	56.3	71.6
	180							73.6	25.8	24.8	95.7	67.1	75.6	99.5	81.4	90.5
	240							83.8	38.1	36.6	99.0	82.0	89.9	99.8	93.0	97.0
25%	60													31.9	3.3	5.6
	120													44.6	9.2	11.5
	180													58.2	14.7	19.0
	240													72.0	24.0	28.6

Note: BJS: Basak *et al*. (2002) test; Vertical: vertical test; L&R: Levy and Roll (2010) test. T is the sample size.

[11] The results corresponding to the 1% and 10% probability levels are given in Tables B2 and B3 in Appendix B, respectively.

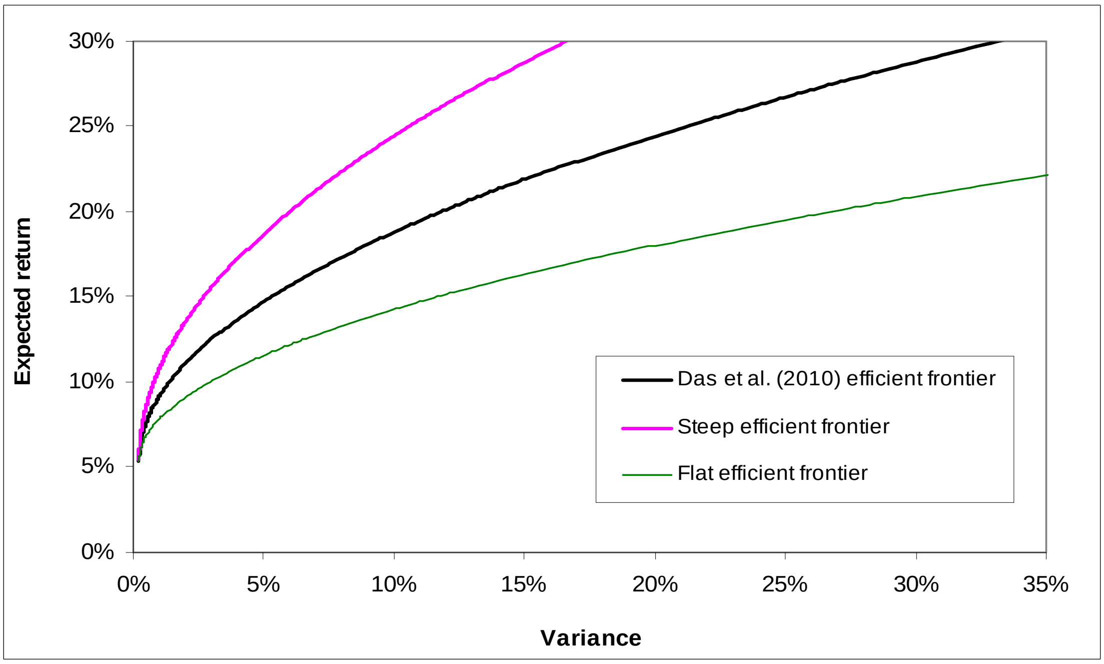
For sample sizes below 180, the power is the lowest for the vertical test, and the highest for the BJS test. The powers of the vertical and L&R tests are also low for the expected returns and variances (20%, 15%) and (25%, 25%). This is likely due to the vertical proximity to the efficient frontier. However, for larger samples, the vertical test outperforms both the BJS and the L&R tests since its size is the lowest for an equivalent power. On the whole, Tables 1 and 2 indicate that the vertical test rejects the null of MV efficiency less frequently than the two other tests.

The differences in power and size between the vertical test and the BJS test might look surprising since both are similar in spirit, namely they are both built from a geometric one-dimensional measure of inefficiency in the MV plane. This counterintuitive result stems from the fact that the standard deviation of the vertical measure of inefficiency is higher than the standard deviation of the horizontal measure used in the BJS test. Indeed, the standard deviations of both tests depend on the absolute values of the weighting loads of the tested-portfolio efficient counterpart (see equations A6 and A7 in Appendix A). However, the efficient "vertical counterparts" are mostly located on the top of the efficient frontier while the efficient "horizontal counterparts" are mostly located at the bottom of the efficient frontier. Since absolute weighting loads are typically higher on the top of the efficient frontier (riskier portfolios are less diversified), the vertical distance is subject to higher standard deviations than the horizontal BJS test. Consequently, the *t*-statistic generally takes lower values for the vertical test than for the BJS test, and hence the former rejects MV efficiency less frequently than the latter. This feature is particularly relevant when short-selling restrictions are imposed (see Best and Grauer, 1991; Green and Hollifield, 1992; Britten-Jones, 1999).

3.3. Robustness Checks on the Slope of the Efficient Frontier

Both the horizontal and vertical measures of portfolio inefficiency are restricted to a single dimension in the MV plane. They are, therefore, sensitive to the slope of the efficient frontier. For this reason, we check the robustness of our previous findings by substantially modifying the slope of the efficient frontier. This is achieved by running simulations under two alternative scenarios for the expected return on the speculative stock (15% and 35% respectively instead of 25%) while keeping all other parameters in Equation (7) unchanged. As Figure 3 shows, the first case (15%) produces a flatter efficient frontier, whereas the second (35%) leads to a steeper MV efficient frontier. The minimum-variance portfolios of the three efficient frontiers still remain very close to each other. As previously, we apply the three efficiency tests to a grid of efficient and non-efficient simulated portfolios.

Figure 3. The three efficient frontiers under consideration



The results are reported in Tables C1 to C4 in Appendix C. They can be summarized as follows. For the flat efficient frontier, the BJS test produces the highest size distortions, while the vertical test exhibits the lowest. Given that the BJS test outperforms the other two tests in terms of power irrespective of the sample size, a reasonable procedure for practical use is to combine the BJS and the vertical tests when the MV efficient frontier is flat. In the case of a steep efficient frontier, the results are similar to those obtained in the benchmark case. The vertical test exhibits the lowest size distortions, and its power strongly increases in comparison to the benchmark case, especially for small samples. On the whole, our results show that the vertical test is preferable when the efficient frontier is steep and samples are large.

4. Is the Market Portfolio Efficient?

In this section, we apply the BJS, the L&R, and the vertical tests of MV efficiency to the capitalization-weighted market portfolio made up of the 100 largest U.S. stocks[12] by market capitalizations as measured on December 31, 2010. The data are monthly returns over the period January 1988 – December 2010 (276 observations). To gauge the sensitivity of our results with respect to the number of available stocks,[13] we also run the tests in stock universes of different sizes ($N = 10, 20, \ldots, 100$).[14] In each case, we select the largest stocks of the sample. For the L&R test we follow the original paper when assessing MV efficiency and use a value of α equal to 0.75. As robustness checks, we also (i) test the MV efficiency for a value α (0.98)—which gives a similar importance to deviations from variance and mean[15]—and (ii) apply the three tests to equally-weighted portfolios.

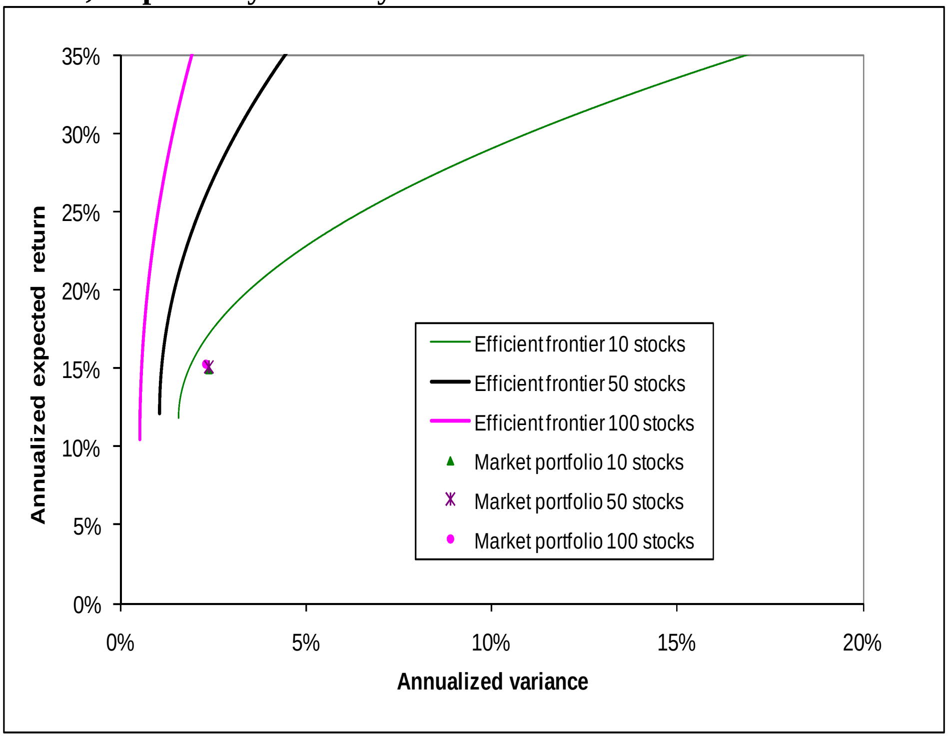
Figure 4 shows the efficient frontiers (without short-selling restrictions) made of 10, 50 and 100 assets, respectively, and the corresponding market portfolios. Noticeably, the MV characteristics of the market portfolio are stable with respect to the number of assets, but the

[12] We selected the 100 largest stocks of the S&P 500 index.
[13] The data are extracted from the Datastream database. Descriptive statistics are given in Appendix D.
[14] In reality, individual investors rarely hold portfolios containing 100 assets (Barber and Odean, 2000; Polkovnichenko, 2005; Goetzmann and Kumar, 2008). The diversification benefits tend to be exhausted once an equity portfolio contains several tens of stocks (Evans and Archer, 1968; Elton and Gruber, 1977; Statman, 1987).
[15] This value is actually very close to the 0.98-value considered in L&R as more realistic than the 0.75 used to test the MV efficiency.

efficient frontier becomes steeper when N increases. In particular, this feature shows that all configurations explored in Section 3 are realistic.

Table 3 summarizes the outcomes of the three tests. Two findings stand out. Firstly, for all sample sizes, both the BJS and the vertical tests reject the null of market portfolio efficiency. Regardless of the number of stocks in the universe, the market portfolio is never MV efficient. Similar results are found for equally-weighted portfolios (see Table 4).

Figure 4. Efficient frontiers and market portfolios for the 10, 50 and 100 largest U.S. stocks, respectively. January 1988 – December 2010



Secondly, for all values of N, the L&R test does not reject market portfolio efficiency for α = 0.75, confirming the findings of L&R.[16] However, for α = 0.98 the L&R test rejects market portfolio efficiency. This indicates that the L&R test is sensitive to the value taken by parameter α. In fact, for α higher than 0.902, MV efficiency is always rejected by the L&R test.

[16] Even though our sample period is longer than in L&R.

Table 3. MV efficiency tests for the capitalization-weighted market portfolio

Nb. of stocks	Annualized Expected Return (in %)	Volatility (in %)	BJS test	Vertical test	L&R test (α = 0.75)	L&R test (α = 0.98)
10	14.84	15.49	-3.11(0.00)	1.28 (0.10)	6.09 (1.00)	161.27 (0.00)
20	15.55	16.36	-4.58 (0.00)	2.14 (0.02)	15.54 (1.00)	579.43 (0.00)
30	14.92	15.63	-4.67 (0.00)	2.32 (0.01)	18.87 (1.00)	773.40 (0.00)
40	15.21	15.64	-5.25 (0.00)	2.94 (0.00)	28.49 (1.00)	1597.15 (0.00)
50	15.05	15.48	-5.54 (0.00)	3.25 (0.00)	37.61 (1.00)	2562.73 (0.00)
60	15.20	15.54	-5.90 (0.00)	3.78 (0.00)	48.73 (1.00)	3357.71 (0.00)
70	15.27	15.40	-6.56 (0.00)	4.46 (0.00)	65.54 (1.00)	3106.69 (0.00)
80	15.33	15.31	-6.53 (0.00)	4.58 (0.00)	76.76 (1.00)	3491.16 (0.00)
90	15.23	15.22	-6.83 (0.00)	4.74 (0.00)	89.71 (1.00)	3542.50 (0.00)
100	15.25	15.22	-7.17 (0.00)	5.05 (0.00)	102.27 (1.00)	4045.07 (0.00)

Coefficient α denotes the MV trade-off in the L&R test statistic. p-values are given in parentheses.

Table 4. MV efficiency tests for the equally-weighted market portfolio

Nb. of stocks	Annualized Expected Returns (in %)	Volatility (in %)	BJS test	Vertical test	L&R test (α = 0.75)	L&R test (α = 0.98)
10	14.29	14.95	-3.22 (0.00)	1.33 (0.09)	6.78 (1.00)	197.70 (0.00)
20	15.34	16.79	-4.56 (0.00)	2.18 (0.01)	15.75 (1.00)	706.71 (0.00)
30	14.32	15.50	-4.54 (0.00)	2.39 (0.01)	19.37 (1.00)	979.52 (0.00)
40	15.17	15.72	-4.99 (0.00)	2.90 (0.00)	28.48 (1.00)	1771.03 (0.00)
50	14.79	15.47	-5.27 (0.00)	3.23 (0.00)	36.90 (1.00)	2681.93 (0.00)
60	15.22	15.76	-5.65 (0.00)	3.75 (0.00)	47.80 (1.00)	3381.66 (0.00)
70	15.39	15.46	-6.14 (0.00)	4.36 (0.00)	64.71 (1.00)	3453.09 (0.00)
80	15.53	15.28	-6.00 (0.00)	4.45 (0.00)	75.95 (1.00)	3938.86 (0.00)
90	15.21	15.13	-6.29 (0.00)	4.60 (0.00)	89.03 (1.00)	4137.95 (0.00)
100	15.30	15.17	-6.68 (0.00)	4.92 (0.00)	102.12 (1.00)	4535.09 (0.00)

Coefficient α denotes the MV trade-off in the L&R test statistic. p-values are given in parentheses.

The large difference in the results depending on whether one considers the vertical and horizontal tests on one side, and the L&R test on the other (the latter depending crucially on the value of α chosen), is striking. Two possible issues in the L&R test might however explain this difference. First, L&R do not acknowledge that μ^* and σ^* are sample-dependent. Indeed, μ^* and σ^* are derived from the minimization of distance $d\left((\mu,\sigma),(\mu,\sigma)^{sam}\right)$ in Eq. (10), where $(\mu,\sigma)^{sam}$ is made of sample parameters. As a consequence, the asymptotic distribution of the test statistic could deviate from the chi-squared distribution. Despite this, the L&R test is applied as if μ^* and σ^* were simple numbers.

Second, to prove the efficiency of the market portfolio, L&R argue that their test should not reject the null for at least one value of α between 0 and 1. To illustrate their point they choose for α a value very close to 1, namely $\alpha = 0.98$. This raises additional issues regarding the power of their likelihood-test. More precisely, the number of degrees of freedom of the chi-square variable used in the L&R test is equal to (2 N + 2) when $0 < \alpha < 1$, but reduces to (N + 2) when $\alpha = 1$. The case $\alpha = 0.98$ is borderline. Indeed, as displayed by Figures E1 and E2 in Appendix E, for $\alpha = 0.98$ the modified asset mean returns change dramatically, whereas standard deviations remain almost unchanged. L&R report a likelihood-ratio test statistic equal to 156.8 (L&R, p. 2472). This number corresponds to the 0.011 fractile of the chi-square distribution with 200 degrees of freedom, but also to the 0.9998 fractile of the chi-square distribution with 100 degrees of freedom. This huge difference suggests that the L&R test has low power for $\alpha = 0.98$, which favors the argument of mean variance efficiency.

On the whole, while the conclusion of the L&R test depends on the trade-off coefficient α, the two other tests unequivocally conclude that the market portfolio is never MV efficient. The validity of the zero-beta CAPM, relying on the efficiency of the market portfolio, is thus strongly called into question. In a nutshell, the fundamental contributions of both Roll (1977) and Ross (1977) remain highly relevant for portfolio management.

5. Conclusion

Our new test of portfolio MV efficiency is based upon the vertical distance of a portfolio from the efficient frontier. While the evidence is mixed for small samples, our test outperforms the previous MV efficiency tests proposed by Basak *et al*. (2002) and Levy and Roll (2010) for large samples since it produces lower size distortions for comparable power. The empirical analysis shows that the L&R test is sensitive to the value taken by the nuisance parameter determining the relative weight assigned to sample-mean changes against standard-deviation changes. Furthermore, both the vertical and horizontal tests are based on intuitive measures in the MV plane and are, therefore, easy to visualize, which makes them more appealing than the L&R test.

The ideally balanced distance in the MV plane remains, however, the orthogonal distance. Even though a test based on this distance is feasible in theory, deriving its closed-form asymptotics could prove challenging. We leave this for further work. Meanwhile, the best alternative for practitioners to test portfolio efficiency is probably the dual approach combining the vertical and horizontal tests. In the final decision, the weight to be allocated to each test could then take into account the curvature of the efficient frontier and may depend on the investor's sensitivity to the risk or return's dimension of his investment.

Both vertical and horizontal MV efficiency tests could of course be improved. Implementing the jackknife-type estimator of the covariance matrix developed by Basak *et al.* (2009) could offer a promising extension since this estimator produces a more accurate covariance matrix than the sample one.

Lastly, our empirical application to the U.S. equity market highlights that the market portfolio is not MV efficient, invalidating the zero-beta CAPM. Consequently, our findings indicate that scepticism on the validity of the CAPM seems to survive the recent rehabilitation attempts made by Levy and Roll (2010).

References

Acharya, V.V., D.M. Gale and T. Yorulmazer. 2011. Rollover Risk and Market Freezes. *Journal of Finance* 66:1177 – 1209.

Barber, B.M. and T. Odean. 2000. Trading Is Hazardous to Your Wealth: The Common Stock Investment Performance of Individual Investors. *Journal of Finance* 55:773-806.

Basak, G., R. Jagannathan, and G. Sun. 2002. A Direct Test for the Mean-Variance Efficiency of a Portfolio. *Journal of Economic Dynamics and Control* 26:1195-1215.

Basak, G., R. Jagannathan, and T. Ma. 2009. Jackknife Estimator for Tracking Errors Variance of Optimal Portfolios. *Management Science* 55:990-1002.

Beaulieu, M.-C., J.-M. Dufour and L. Khalaf. 2008. Finite Sample Identification-Robust Inference for Unobservable Zero-Beta Rates and Portfolio Efficiency with Non-Gaussian Distributions. Technical report, Mc Gill University, Université Laval and Carleton University.

Best, M.J. and R.R. Grauer. 1991. On the Sensitivity of Mean-Variance Efficient Portfolios to Changes in Asset Means: Some Analytical and Computational Results. *Review of Financial Studies* 4:315-342.

Black, F. 1972. Capital Market Equilibrium with Restricted Borrowing. *Journal of Business* 45:444-454.

Brennan, M. 1971. Capital Market Equilibrium with Divergent Borrowing and Lending Rates. *Journal of Financial and Quantitative Analysis* 6:1197-1205.

Britten-Jones, M.J. 1999. The Sampling Error in Estimates of Mean-Variance Efficient Portfolio Weights. *Journal of Finance* 54:655-671.

Bruche, M. and J. Suarez. 2010. Deposit Insurance and Money Market Freezes. *Journal of Monetary Economics* 57:45-61.

Das, S., H. Markowitz, J. Scheid and M. Statman. 2010. Portfolio Optimization with Mental Accounts. *Journal of Financial and Quantitative Analysis* 45:311-334.

De Roon, F.A. and T.E. Nijman. 2001. Testing for Mean-Variance Spanning: A Survey. *Journal of Empirical Finance* 8:111-155.

Drut, B. 2010. Sovereign Bonds and Socially Responsible Investment. *Journal of Business Ethics* 92:131-145.

Ehling, P. and S.B. Ramos. 2006. Geographic versus Industry Diversification: Constraints Matter. *Journal of Empirical Finance* 13:396-416.

Elton, E.J. and M.J. Gruber. 1977. Risk Reduction and Portfolio Size: An Analytical Solution. *Journal of Business* 50:415-437.

Evans, J.L. and S.H. Archer. 1968. Diversification and the Reduction of Dispersion: An Empirical Analysis. *Journal of Finance* 23:761-767.

Gerard, B., P. Hillion, F.A. de Roon, and E. Eiling. 2007. International Portfolio Diversification: Currency, Industry and Country Effects Revisited. Working paper available at doi:10.2139/ssrn.302353.

Gibbons, M.R. 1982. Multivariate Tests of Financial Models: A New Approach. *Journal of Financial Economics* 10:3-27.

Gibbons, M.R., S.A. Ross and J. Shanken. 1989. A Test of the Efficiency of a Given Portfolio. *Econometrica* 57:1121-1152.

Goetzmann, W.N. and A. Kumar. 2008. Equity Portfolio Diversification. *Review of Finance,* 12:433-463.

Gourieroux, C., O. Scaillet, and A. Szafarz. 1997. *Econométrie de la Finance : Approches Historiques*. Paris: Economica.

Green, R.C. and B. Hollifield. 1992. When Will Mean-Variance Efficient Portfolios Be Well Diversified? *Journal of Finance* 47:1785-1809.

Jobson, J.D. and B. Korkie. 1982. Potential Performance and Tests of Portfolio Efficiency. *Journal of Financial Economics* 10:433-466.

Jorion, P. 2003. Portfolio Optimization with Tracking-Error Constraints. *Financial Analyst Journal* 59: 70-82.

Kandel, S. 1984. The Likelihood Ratio Test Statistic of Mean-Variance Efficiency without a Riskless Asset. *Journal of Financial Economics* 13:575-592.

Kandel, S. 1986. The Geometry of the Likelihood Estimator of the Zero-Beta Return. *Journal of Finance* 41:339-346.

Kandel, S. and R.F. Stambaugh. 1995. Portfolio Inefficiency and the Cross-Section of Expected Returns. *Journal of Finance* 50:157-184.

Krishnamurthy, A. 2010. The Financial Meltdown: Data and Diagnoses. Working Paper, Northwestern University.

Levy, M. and R. Roll. 2010. The Market Portfolio May Be Mean/Variance Efficient After All. *Review of Financial Studies* 23:2464-2491.

Li, K., S. Asani and Z. Wang. 2003. Diversification Benefits of Emerging Markets Subject to Portfolio Constraints. *Journal of Empirical Finance* 10:57-80.

Lintner, J. 1965. Security Prices, Risk, and the Maximal Gains from Diversification. *Journal of Finance* 20:587-615.

MacKinlay, A.C. and M.P. Richardson. 1991. Using Generalized Method of Moments to Test Mean-Variance Efficiency. *Journal of Finance* 46:511-527.

Markowitz, H. 1952. Portfolio Selection. *Journal of Finance* 7:77-91.

Markowitz, H. 1959. *Portfolio Selection: Efficient Diversification of Investments*. New York: John Wiley.

Michaud, R.O. 1989. The Markowitz Optimization Enigma: Is 'Optimized' Optimal? *Financial Analysts Journal* 45:31-42.

Polkovnichenko, V. 2005. Household Portfolio Diversification: A Case for Rank-Dependent Preferences. *Review of Financial Studies* 18:1467-1502.

Roll, R. 1977. A Critique of the Asset Pricing Theory's Tests. Part I: On Past and Potential Testability of the Theory. *Journal of Financial Economics* 4:129-176.

Roll, R. 1992. A Mean/Variance Analysis of Tracking Error. *Journal of Portfolio Management* 18: 13-22.

Ross, S.A. 1977. The Capital Asset Pricing Model (CAPM), Short-Sale Restrictions and Related Issues. *Journal of Finance* 19:425-442.

Shanken, J. 1985. Multivariate Tests of the Zero-Beta CAPM. *Journal of Financial Economics* 14:327-348.

Shanken, J. 1986. Testing Portfolio Efficiency when the Zero-Beta Rate is Unknown: A Note. *Journal of Finance* 41:269-276.

Sharpe, W.F. 1964. Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. *Journal of Finance* 19:425-442.

Statman, M. 1987. How Many Stocks Make a Diversified Portfolio? *Journal of Financial and Quantitative Analysis* 22:353-363.

Topaloglou, N. and O. Scaillet. 2010. Testing for Stochastic Dominance. *Journal of Business and Economic Statistics* 28:169-180.

Velu, R. and G. Zhou. 1999. Testing Multi-Beta Asset Pricing Models. *Journal of Empirical Finance* 6:219-241.

Wang, Z. 1998. Efficiency Loss and Constraints on Portfolio Holdings. *Journal of Financial Economics* 48:359-375.

Wasserman, L. 2004. *All of Statistics: A Concise Course in Statistical Inference*. New York: Springer.

Zhou, G. 1991. Small Sample Tests of Portfolio Efficiency. *Journal of Financial Economics* 30:165-191.

Appendix A: Proof of Proposition 1

We first derive the asymptotic distribution of the vertical distance, $\hat{\theta}$, defined in Equation (5) in the case where short-selling is forbidden. At the end of this Appendix, we extend the results to the case where short-selling is allowed

Let x be a k-dimensional vector, and denote $x^{(i)} = (x_i, x_{i+1}, ..., x_k)'$. Consider a symmetric matrix B of order k, and $B = [B_1 : B_2 : ... : B_k]$ where B_i is the i^{th} column of B. Let $vec(B)$ be the stacked vector of the columns of B:

$$vec(B) = (B_1^{(1)'}, B_2^{(2)'}, \dots, B_k^{(k)'})'$$

Next, let $\overline{V}$ be the vector formed by stacking the sample mean of R_t, the elements of $\text{cov}(R_t)$, the sample mean of r_t, and the sample variance of r_t:

$$\overline{V} = (\hat{\mu}', (vec(\hat{\Sigma}))', \hat{\beta}, \hat{v^2})'$$

Vector $\overline{V}$ thus summarizes the first and second moments of the sample returns. Similarly to BJS (2002), we express vector $\overline{V}$ as a function of the sample non-central first and second moments of R_t and r_t. The transformed vector, U_t, is defined by:

$$U_t = (R_t^{'}, (vec(R_t R_t'))', r_t, r_t^2)' = (R_t^{'}, Y_t', r_t, w_t)'$$

and its sample mean, $\overline{U}$, is:

$$\overline{U} = \frac{1}{T} \sum_{t=1}^{T} U_t = \frac{1}{T} \sum_{t=1}^{T} [R_t' : Y_t' : r_t : w_t]'$$

Let $g(.)$ denote the function that maps vector $\overline{U}$ to vector $\overline{V}$:

$$g(\overline{U}) = g\begin{pmatrix} \hat{\mu} \\ \overline{Y} \\ \hat{\beta} \\ \overline{w} \end{pmatrix} = \begin{pmatrix} \hat{\mu} \\ \overline{Y} - vec(\hat{\mu}\hat{\mu}') \\ \hat{\beta} \\ \overline{w} - \hat{\beta}^2 \end{pmatrix} = \begin{pmatrix} \hat{\mu} \\ vec(\hat{\Sigma}) \\ \hat{\beta} \\ \hat{v^2} \end{pmatrix} = \overline{V}$$

By applying the delta method, when T tends to the infinite, we have:

$$\sqrt{T}\left(\bar{V}-V\right)=\sqrt{T}\left(g(\bar{U})-g(\alpha)\right)\rightarrow N(0,\Delta) \tag{A1}$$

where
$$\Delta = D\Lambda_0 D' \tag{A2}$$

$D=\left(\dfrac{\partial g_i}{\partial U_j}\right)$ and Λ_0 being the covariance matrix of U_t, and from BJS (2002, p. 1208):

$$D=\frac{\partial g}{\partial x}\Big|_{x=\bar{U}_T}=\begin{pmatrix} I_p & 0_{p\times pv} & 0_{p\times 2} \\ K_1 & & \\ \vdots & I_{pv} & 0_{pv\times 2} \\ K_p & & \\ 0_{2\times p} & 0_{2\times pv} & \begin{pmatrix} 1:0 \\ -2\hat{\beta}:1 \end{pmatrix} \end{pmatrix} \tag{A3}$$

Where $pv=\frac{p(p+1)}{2}$; $K_i=-\left[0_{(p-i+1)\times(i-1)}:\hat{\mu}^{(i)}:0_{(p-i+1)\times(p-i)}\right]-\hat{\mu}_i\left[0_{(p-i+1)\times(i-1)}:I_{p-i+1}\right]$; $\hat{\mu}_i$ stands for the i^{th} element of $\hat{\mu}$, and I_Z stands for the identity matrix of rank Z.

The asymptotic distribution of vector $\bar{V}$ is given by (A1). Let us now move to the vertical distance, $\hat{\theta}$, which is a differentiable function of vector $\bar{V}$. Consequently, the delta method establishes that the asymptotic variance ϕ^2 of $\hat{\theta}$ is $\lim_{T\to\infty}\frac{\partial\hat{\theta}}{\partial\bar{V}}\Delta\frac{\partial\hat{\theta}}{\partial\bar{V}}$, where derivative $\frac{\partial\hat{\theta}}{\partial\bar{V}}$ needs to be computed. With this aim, we express that $\hat{\theta}$ minimizes the following Lagrangian function:

$$l=\hat{\beta}-\omega'\hat{\mu}+\delta_1(\omega'\iota-1)+\delta_2(\omega'\hat{\Sigma}\omega-\hat{v}^2)-\varphi'\omega \tag{A4}$$

By differentiation, we have:

$$\frac{\partial\hat{\theta}}{\partial\bar{V}}=\frac{\partial l}{\partial\bar{V}}=(-\hat{\mu}'-\varphi'+\delta_1\iota'+\delta_2 2\,\omega'\hat{\Sigma})\frac{\partial\omega}{\partial\bar{V}}$$

$$+\delta_2\left[0_{1\times p}:(\omega_1{}^2:2\omega_1\omega_2:2\omega_1\omega_3:\ldots:2\omega_1\omega_p:\omega_2{}^2:\ldots:\omega_p{}^2):0:-1\right]+\left(-\omega':0_{1\times pv}:1:0\right) \tag{A5}$$

From the first order condition applied to (A4), we obtain:

$$\frac{\partial l}{\partial \omega} = 0_{p\times 1} = -\hat{\mu} - v' + \delta_1 \iota + \delta_2 (2\hat{\Sigma}\omega)$$

And consequently:

$$\frac{\partial \hat{\theta}}{\partial \overline{V}} = \left(-\omega' : 0_{1\times pv} : 1 : 0\right) + \delta_2 \left[0_{1\times p} : (\omega_1{}^2 : 2\omega_1\omega_2 : 2\omega_1\omega_3 : \ldots : 2\omega_1\omega_p : \omega_2{}^2 : \ldots : \omega_p{}^2) : 0 : -1\right] \qquad \text{(A6)}$$

Combining the results in (A1), (A4) and (A6), we obtain the asymptotic variance ϕ^2 of the vertical distance $\hat{\theta}$:

$$\phi^2 = \lim_{T\to\infty} \frac{\partial \hat{\theta}}{\partial \overline{V}} \Delta \frac{\partial \hat{\theta}}{\partial \overline{V}}$$

(A7)

When there are no short-selling restrictions, the efficient frontier is modified because the sole constraint applied to ω is that its components add up to one. Let $\hat{\theta}*$ denote the vertical distance in this case. The modified Lagrangian function is:

$$l* = \hat{\beta} - \omega' \hat{\mu} + \delta_1 (\omega' \iota - 1) + \delta_2 (\omega' \hat{\Sigma} \omega - \hat{v^2}) \qquad \text{(A8)}$$

By differentiating both sides of (A7), we get:

$$\frac{\partial l*}{\partial \overline{V}} = \left(-\omega' : 0_{1\times pv} : 1 : 0\right) + \delta_2 \left[0_{1\times p} : (\omega_1{}^2 : 2\omega_1\omega_2 : 2\omega_1\omega_3 : \ldots : 2\omega_1\omega_p : \omega_2{}^2 : \ldots : \omega_p{}^2) : 0 : -1\right] \qquad \text{(A9)}$$

Substituting $\frac{\partial l*}{\partial \overline{V}}$ in (A8) by $\frac{\partial l*}{\partial \overline{V}}$ from (A5) gives the asymptotic variance $\phi^2 *$ of the vertical distance $\hat{\theta}*$ when there are no short-selling restrictions. Its expression stands as:

$$\phi^2 * = \lim_{T\to\infty} \frac{\partial \hat{\theta}*}{\partial \overline{V}} \Delta \frac{\partial \hat{\theta}*}{\partial \overline{V}}$$

(A10)

Lastly, as mentioned by BJS, bringing all the variables in the same scale by imposing constraint $\sqrt{\omega' \hat{\Sigma} \omega} = \sqrt{\hat{v}^2}$ would likely improve the test efficiency and reduce the corresponding bias.

Appendix B: Rejection Frequencies at the 1% and 10% Probability Levels

Table B1. Rejection frequencies (in percent) at the 1% and 10% probability levels for the efficient portfolios

		1% probability error			10% probability error		
	T	**BJS**	**Vertical**	**L&R**	**BJS**	**Vertical**	**L&R**
Expected return 10%	60	1.7	0.0	2.2	16.7	2.2	4.7
	120	0.9	0.0	0.6	12.3	1.3	3.0
	180	0.6	0.0	0.7	12.8	1.3	2.6
	240	0.5	0.0	0.4	11.2	1.2	2.1
15%	60	2.2	0.0	1.7	13.6	2.3	4.0
	120	1.6	0.0	0.7	14.2	1.5	2.7
	180	1.3	0.0	0.5	12.4	1.6	2.3
	240	0.9	0.0	0.4	12.1	1.0	1.8
20%	60	2.4	0.0	1.7	17.8	2.3	4.1
	120	1.0	0.0	0.6	14.5	1.3	2.9
	180	0.8	0.0	0.6	13.7	1.3	2.3
	240	0.8	0.0	0.4	12.1	1.2	2.1
25%	60	2.2	0.0	1.4	14.1	2.4	4.1
	120	1.5	0.0	0.7	14.3	1.6	2.9
	180	1.3	0.0	0.5	12.5	1.4	2.2
	240	0.9	0.0	0.0	12.2	1.1	0.0

Note: BJS: Basak *et al*. (2002) test; Vertical: vertical test; L&R: Levy and Roll (2010) test. *T* denotes the sample size.

Table B2. Rejection frequencies (in percent) at the 1% probability level for the inefficient portfolios

		Variance														
		5%			10%			15%			20%			25%		
	T	**BJS**	**Vertical**	**L&R**	**BJS**	**Vertical**	**L&R**	**BJS**	**Vertical**	**L&R**	**BJS**	**Vertical**	**L&R**	**BJS**	**Vertical**	**L&R**
Expected return 10%	60	78.5	14.7	55.3	86.8	24.4	65.8	92.0	32.4	70.4	92.7	35.7	73.2	95.3	38.8	75.4
	120	96.8	44.1	87.6	99.5	65.4	92.6	99.6	71.1	92.8	99.7	75.0	93.2	99.5	76.5	91.4
	180	99.7	75.2	97.7	99.9	89.2	99.0	100.0	93.0	99.0	99.9	93.2	98.5	100.0	95.6	98.5
	240	99.9	91.1	99.7	100.0	97.9	99.5	100.0	98.5	100.0	100.0	99.0	100.0	100.0	98.7	99.8
15%	60				49.8	3.2	23.8	68.1	9.0	42.4	78.3	18.0	55.8	84.8	17.7	61.2
	120				76.1	13.7	51.6	92.2	30.7	77.3	96.2	42.9	86.5	98.4	51.2	89.3
	180				91.1	31.9	75.2	98.0	59.2	95.1	99.6	75.2	97.1	99.8	79.2	98.7
	240				96.8	47.5	89.0	99.7	79.6	98.5	100.0	90.5	99.3	100.0	96.0	99.9
20%	60							17.1	0.7	3.1	41.9	2.5	16.3	56.4	4.9	31.0
	120							32.6	0.7	5.0	64.6	8.3	39.3	81.5	18.6	56.4
	180							46.3	3.1	12.1	83.7	19.1	63.6	95.0	40.9	83.4
	240							59.6	6.5	19.4	94.3	38.2	80.0	98.6	60.5	94.9
25%	60													13.9	0.1	3.6
	120													20.4	0.8	4.9
	180													31.7	0.9	9.0
	240													44.0	2.0	14.2

Note: BJS: Basak *et al*. (2002) test; Vertical: vertical test; L&R: Levy and Roll (2010) test. *T* denotes the sample size.

Table B3. Rejection frequencies (in percent) at the 10% probability level for the inefficient portfolios

		Variance														
		5%			10%			15%			20%			25%		
Expected return	T	BJS	Vertical	L&R	BJS	Vertical	L&R	BJS	Vertical	L&R	BJS	Vertical	L&R	BJS	Vertical	L&R
10%	60	94.7	70.6	71.8	96.8	79.5	79.7	98.8	84.5	80.7	98.6	86.8	83.6	98.9	87.9	83.1
	120	99.9	95.1	95.6	100.0	98.7	98.1	100.0	98.7	97.9	100.0	99.2	96.9	99.9	98.9	97.6
	180	100.0	99.4	99.6	100.0	100.0	99.8	100.0	100.0	99.8	100.0	99.7	99.6	100.0	100.0	99.9
	240	100.0	99.9	99.9	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
15%	60				81.3	45.0	42.9	91.5	60.8	63.0	93.2	69.9	71.3	96.2	76.0	76.4
	120				96.5	74.8	72.4	99.6	89.8	90.8	99.8	94.0	94.9	99.8	96.2	97.2
	180				99.6	90.4	90.1	100.0	97.7	98.2	100.0	99.6	99.4	100.0	99.5	99.6
	240				99.9	97.0	96.4	100.0	99.3	99.6	100.0	99.9	99.9	100.0	100.0	100.0
20%	60							50.3	15.7	8.8	75.4	37.7	34.0	83.3	51.8	51.0
	120							71.3	35.1	19.4	91.0	66.0	61.8	96.7	77.5	78.8
	180							84.3	48.7	33.5	98.1	84.8	82.8	99.9	93.9	93.5
	240							92.0	63.9	46.2	99.8	95.5	94.0	99.8	98.4	98.4
25%	60													43.7	14.0	7.8
	120													59.8	22.8	15.4
	180													72.3	34.6	25.0
	240													82.9	49.2	37.1

Note: BJS: Basak *et al*. (2002) test; Vertical: vertical test; L&R: Levy and Roll (2010) test. *T* denotes the sample size.

Appendix C: Robustness Checks

Table C1. Flat efficient frontier. Rejection frequencies (in percent) at the 5% probability level for the efficient portfolios

Expected return		T	BJS	Vertical	L&R
	10%	60	14.8	0.6	5.9
		120	10.0	0.2	2.7
		180	8.3	0.1	1.2
		240	8.9	0.3	1.2
	15%	60	15.5	0.7	3.9
		120	10.7	0.5	2.4
		180	9.8	0.1	1.5
		240	8.6	0.1	0.9
	20%	60	16.2	1.0	6.5
		120	11.4	0.5	2.4
		180	9.7	0.3	2.0
		240	9.5	0.6	1.4
	25%	60	15.1	0.5	4.3
		120	11.3	0.2	2.4
		180	9.8	0.3	2.1
		240	8.8	0.1	0.6

Note: BJS: Basak *et al*. (2002) test; Vertical: vertical test; L&R: Levy and Roll (2010) test. *T* denotes the sample size.

Table C2. Flat efficient frontier. Rejection frequencies (in percent) at the 5% probability level for the inefficient portfolios

			Variance														
			5%			10%			15%			20%			25%		
		T	BJS	Vertical	L&R	BJS	Vertical	L&R	BJS	Vertical	L&R	BJS	Vertical	L&R	BJS	Vertical	L&R
Expected return	10%	60	50.8	10.6	24.2	73.1	19.5	21.3	76.7	28.2	30.2	79.4	32.8	35.4	81.3	33.3	37.9
		120	67.7	16.1	20.7	88.3	41.2	30.2	94.1	53.2	43.9	95.2	59.6	51.3	95.1	59.1	53.0
		180	81.2	30.4	31.3	95.9	63.5	51.2	98.4	70.8	60.6	99.0	80.1	71.5	99.4	79.3	71.0
		240	87.6	41.7	42.6	97.9	76.8	67.1	99.3	83.6	76.5	99.8	89.7	84.4	99.9	91.4	85.2
	15%	60				14.5	0.5	3.5	37.5	3.3	15.9	48.7	9.6	15.9	58.5	12.9	15.6
		120				11.9	0.3	2.0	44.8	6.9	19.2	67.8	17.9	21.3	77.2	23.9	15.0
		180				9.7	0.3	1.6	56.3	10.8	21.8	78.9	28.9	31.2	88.4	41.6	27.4
		240				9.5	0.1	0.9	60.5	13.4	25.3	86.5	38.2	39.6	94.8	54.3	41.0

Note: BJS: Basak *et al*. (2002) test; Vertical: vertical test; L&R: Levy and Roll (2010) test. *T* denotes the sample size.

Table C3. Steep efficient frontier. Rejection frequencies (in percent) at the 5% probability level for the efficient portfolios

		T	BJS	Vertical	L&R
Expected return	10%	60	3.0	0.2	4.2
		120	2.1	0.2	3.5
		180	2.6	0.3	3.1
		240	1.0	0.0	1.9
	15%	60	4.1	0.3	5.3
		120	3.5	0.3	3.4
		180	3.5	0.3	3.3
		240	3.6	0.0	3.1
	20%	60	3.6	0.1	5.3
		120	4.5	0.3	4.1
		180	3.6	0.0	4.1
		240	2.5	0.3	2.5
	25%	60	4.2	0.5	4.1
		120	3.4	0.2	3.3
		180	3.4	0.4	3.2
		240	2.6	0.0	2.2

Note: BJS: Basak *et al*. (2002) test; Vertical: vertical test; L&R: Levy and Roll (2010) test. *T* denotes the sample size.

Table C4. Steep efficient frontier. Rejection frequencies (in percent) at the 5% probability level for the inefficient portfolios

			Variance													
			5%			10%			15%			20%			25%	
		T	BJS	Vertical	L&R	BJS	Vertical	L&R	BJS	Vertical	L&R	BJS	Vertical	L&R	BJS	Vertical	L&R
Expected return	10%	60	99.7	91.2	98.0	99.6	96.1	98.2	100.0	95.2	98.3	99.9	96.7	97.5	99.6	96.1	98.2
		120	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.9	100.0	100.0	100.0	100.0
		180	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
		240	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	15%	60	85.4	44.6	78.2	97.5	76.8	94.8	99.3	88.0	97.8	99.9	92.1	98.4	99.6	89.5	98.5
		120	98.8	81.3	96.7	100.0	99.2	100.0	100.0	99.8	100.0	100.0	99.8	100.0	100.0	99.9	100.0
		180	99.7	97.2	99.6	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
		240	100.0	99.7	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	20%	60				78.8	37.3	68.8	92.2	61.9	88.2	97.7	74.1	93.9	97.9	80.5	94.7
		120				96.2	71.9	93.9	99.9	94.2	99.6	100.0	97.4	99.9	99.9	99.0	99.9
		180				99.4	93.0	99.4	100.0	99.7	100.0	100.0	99.9	100.0	100.0	100.0	100.0
		240				100.0	98.0	99.6	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	25%	60							57.9	19.2	46.1	81.2	40.6	73.5	92.5	61.6	87.9
		120							87.5	53.0	79.9	98.2	83.8	97.3	99.5	94.0	99.6
		180							96.2	75.7	94.4	99.9	96.5	99.7	100.0	99.7	99.9
		240							99.1	92.8	98.5	100.0	99.5	100.0	100.0	100.0	100.0

Note: BJS: Basak *et al*. (2002) test; Vertical: vertical test; L&R: Levy and Roll (2010) test. *T* denotes the sample size.

Appendix D: Descriptive Statistics for the Considered U.S. stocks

Table D1. Descriptive statistics of the stocks' monthly returns over the period January 1988 – December 2010

Company	Annualized mean return (in %)	Annualized volatility (in %)	Market capitalization in billion USD as of December 31, 2010
EXXON MOBIL	9.8	16.1	368.7
APPLE	26.9	47.8	295.9
MICROSOFT	24.6	34.6	238.8
GENERAL ELECTRIC	10.3	25.9	194.9
WAL MART STORES	14.8	23.5	192.1
CHEVRON	11.1	19.7	183.6
INTERNATIONAL BUS.MCHS.	11.3	28.6	182.3
PROCTER & GAMBLE	13.2	20.8	180.1
AT&T	7.7	23.8	173.6
JOHNSON & JOHNSON	13.2	20.5	169.9
JP MORGAN CHASE & CO.	13.9	34.9	165.8
WELLS FARGO & CO	17.1	29.9	162.7
ORACLE	34.6	49.0	158.1
COCA COLA	14.0	22.0	152.7
PFIZER	11.9	24.4	140.3
CITIGROUP	12.7	41.6	137.4
BANK OF AMERICA	12.0	39.4	134.5
INTEL	22.3	39.3	117.3
SCHLUMBERGER	15.2	30.1	113.9
MERCK & CO.	10.1	26.5	111.0
PEPSICO	13.4	21.3	103.5
VERIZON COMMUNICATIONS	6.1	23.6	101.1
CONOCOPHILLIPS	13.1	25.2	100.1
HEWLETT-PACKARD	15.1	35.3	92.2
MCDONALDS	14.0	22.4	81.1
OCCIDENTAL PTL.	12.4	26.3	79.7
ABBOTT LABORATORIES	11.3	20.0	74.1
UNITED TECHNOLOGIES	15.2	23.9	72.7
WALT DISNEY	12.5	26.3	71.0
3M	9.8	20.4	61.7
CATERPILLAR	16.0	31.1	59.4
HOME DEPOT	22.0	29.6	57.5
FORD MOTOR	12.8	46.3	57.1
AMGEN	25.4	35.6	51.9
US BANCORP	15.7	29.2	51.7
AMERICAN EXPRESS	13.2	33.0	51.7
ALTRIA GROUP	15.4	26.7	51.4
BOEING	12.2	28.0	47.9
CVS CAREMARK	10.8	26.2	47.2
EMC	33.5	52.1	47.2
UNION PACIFIC	12.8	23.7	45.7
COMCAST 'A'	15.2	32.8	45.7

Company	Annualized mean return (in %)	Annualized volatility (in %)	Market capitalization in billion USD as of December 31, 2010
E I DU PONT DE NEMOURS	8.7	24.9	45.5
BRISTOL MYERS SQUIBB	6.8	23.2	45.3
APACHE	21.4	35.3	43.5
EMERSON ELECTRIC	10.9	22.1	43.0
TARGET	17.2	28.1	42.6
HONEYWELL INTL.	13.1	30.2	41.5
ELI LILLY	9.3	27.1	40.4
MEDTRONIC	17.7	26.0	39.8
UNITEDHEALTH GP.	30.6	35.1	39.7
DOW CHEMICAL	8.6	35.4	39.6
COLGATE-PALM.	14.5	23.2	38.8
TEXAS INSTS.	19.0	41.8	38.2
ANADARKO PETROLEUM	16.6	34.7	37.7
BANK OF NEW YORK MELLON	13.6	30.9	37.5
HALLIBURTON	15.0	37.5	37.1
WALGREEN	16.9	26.3	35.9
DEERE	15.8	29.5	35.1
LOWE'S COMPANIES	22.7	35.7	34.6
DEVON ENERGY	25.5	39.3	33.9
NIKE 'B'	24.8	33.6	33.2
SOUTHERN	8.8	17.5	32.1
PNC FINL.SVS.GP.	8.8	29.1	31.9
DANAHER	23.1	28.5	30.8
CORNING	19.7	52.0	30.2
NEWMONT MINING	10.9	38.9	29.9
BAXTER INTL.	10.3	24.8	29.5
FEDEX	14.6	31.0	29.3
CARNIVAL	17.6	34.6	28.0
CELGENE	37.1	68.4	27.8
EXELON	8.6	22.9	27.5
GENERAL DYNAMICS	13.8	26.1	26.8
AFLAC	20.1	32.1	26.6
ILLINOIS TOOL WORKS	14.2	24.5	26.5
JOHNSON CONTROLS	16.4	29.7	25.9
HESS	13.6	28.9	25.8
KIMBERLY-CLARK	9.1	20.2	25.7
TRAVELERS COS.	9.8	25.9	25.6
FRANKLIN RESOURCES	22.6	34.2	25.4
DOMINION RES.	5.9	17.3	25.2
BAKER HUGHES	12.2	35.7	24.7
CSX	13.0	26.8	24.2
DUKE ENERGY	6.1	20.4	23.6
STATE STREET	17.6	32.8	23.3
NORFOLK SOUTHERN	11.9	26.8	22.8
AUTOMATIC DATA PROC.	12.2	21.5	22.8
GENERAL MILLS	10.1	18.3	22.6
THERMO FISHER SCIENTIFIC	17.3	30.9	22.0
CUMMINS	20.4	39.0	21.8

Company	Annualized mean return (in %)	Annualized volatility (in %)	Market capitalization in billion USD as of December 31, 2010
NEXTERA ENERGY	6.9	18.5	21.6
STRYKER	23.5	32.6	21.3
MOTOROLA SOLUTIONS	11.5	36.9	21.3
PACCAR	18.3	31.8	20.9
CHARLES SCHWAB	30.7	45.3	20.4
PREC.CASTPARTS	20.2	34.6	19.9
AIR PRDS.& CHEMS.	13.0	26.4	19.5
ARCHER-DANLS.-MIDL.	12.2	27.9	19.2
BECTON DICKINSON	13.6	24.0	19.1
NORTHROP GRUMMAN	10.6	30.0	18.9

Appendix E: Sample versus modified parameters in the L&R paper

Figure E1. Sample vs. modified expected returns in the L&R paper

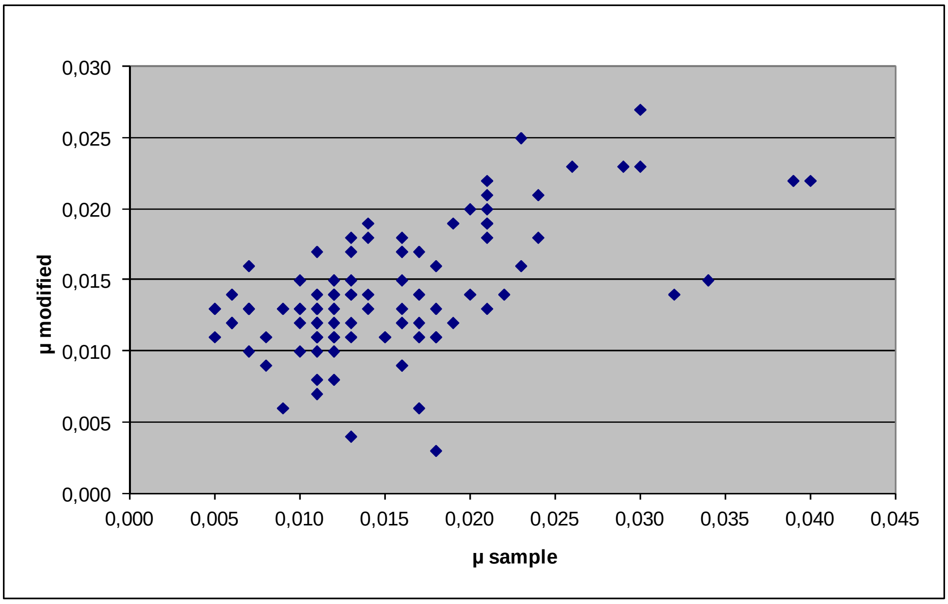
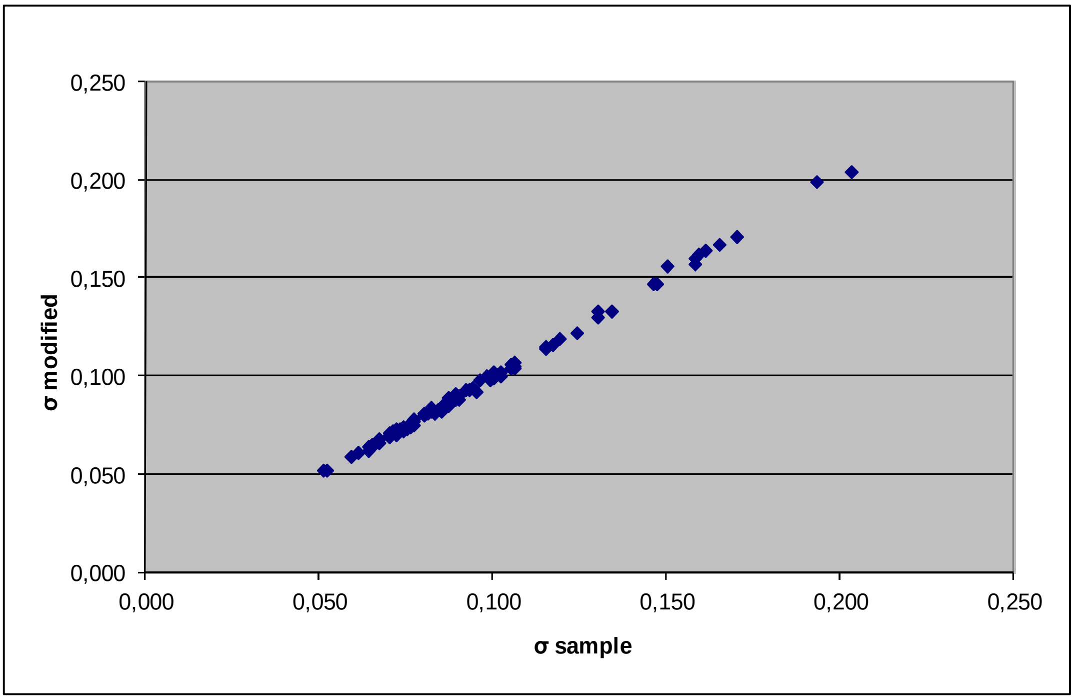


Figure E2. Sample vs. modified standard deviations in the L&R paper



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Amundi Working Paper

WP-019-2012

January 2012

Revised: July 2012