The Reactive Covariance Model and its implications in asset allocation

Eduardo Abi Jaber, Quantitative Analyst – ENSAE ParisTech

Dave Benichou, Portfolio Manager – Amundi

Hassan Malongo, Quantitative Analyst – Amundi

The Reactive Covariance Model and its implications in asset allocation

Eduardo Abi Jaber

Quantitative Analyst – ENSAE ParisTech

Dave Benichou

Portfolio Manager – Amundi

Hassan Malongo, PhD

Quantitative Analyst – Fund Management and Research Team – Amundi Hong Kong

hassan.malongo@hk.amundi.com

About the author

Eduardo Abi Jaber, Quantitative Analyst, ENSAE ParisTech

Graduated from both ENSAE ParisTech (school of economics, statistics and finance) and Paris Diderot University, Eduardo Abi Jaber is a research analyst. He was a key member of a workgroup supervised by Amundi on volatility modeling and its implications in portfolio optimization and risk management.

Dave Benichou, Portfolio Manager – Amundi

Dave Benichou started his career in Portfolio Management at BNP Paribas and HSBC in 2000. He then joined Systeia Capital Management in June 2001 (part of Amundi Group) as an Equity Analyst and Trader for the Systematic and Statistical Arbitrage Desk. In 2005, he expanded his career at Systeia as a Portfolio Manager where he implemented Style Arbitrage Strategies. In July 2008, Dave joined Amundi Paris to manage an Equity Market Neutral Fund. He was nominated in 2003 and 2007 in EuroHedge awards, and won the "Best UCITS fund" HFR award in 2011. Dave Benichou received a postgraduate degree in Banking and Finance from Sorbonne University in 2001 following his Master's Degree in Mathematics and Economics from Nice Sophia-Antipolis.

Hassan Malongo, PhD - Quantitative Analyst, Fund Management and Research Team – Amundi Hong Kong

Hassan Malongo joined Amundi Hong Kong in July 2014 as quantitative analyst within the investment team. Prior to this, he worked as a research analyst within the investor research center team at Amundi Paris since 2010 after being a quantitative analyst within the equity arbitrage team at Crédit Agricole Asset Management (CAAM). He is both graduated from Ensae ParisTech (school of economics, statistics and finance) and Pierre & Marie Curie University and he holds a PhD in applied mathematics from the University of Paris-Dauphine. He is the author of a number of scientific articles published in academic and practitioners' journals and his research interests focus particularly on portfolio construction, multi-assets portfolio dependence modeling, statistical arbitrage and risk management.

Abstract

3

Accurate estimations of volatility and correlation risk represent crucial inputs in terms of investment decisions. This article presents a new way to capture the portfolio dependence by introducing a new covariance estimator called the reactive covariance model. This new model is easy to implement and comes from the generalization to the multivariate framework of the reactive volatility model introduced by Valeyre *et al.* (2013). We examine the properties of this new covariance estimator and present its attractive features by exploring its ability to capture the dominant factors that create changes in the correlation structure of asset returns. By comparing our model with other traditional existing covariance models, we finally examine and present the advantages of using this new model in multi-asset portfolio construction.

Key words: Covariance matrix, investment strategy, asset allocation, portfolio choice, risk parity.

JEL Classification: G01, G11, G12, G14, G15

1. Introduction

Building a realistic covariance model that is both statistically relevant and consistent with empirical properties of asset returns is still a key issue for academics and practitioners. Indeed, the stylized facts of financial assets such as fat tails, volatility clustering, leverage effect or conditional heteroskedasticity are not easy to capture and that explains why modelling the joint behaviour of asset returns remains a big concern in asset allocation for instance.

Considering the univariate framework, a lot of specifications have been proposed to capture these stylized facts in the academic literature. Since the seminal paper of Bollerslev (1987)[1], the standard GARCH model has been one of the most commonly used models in empirical studies to estimate the conditional volatility, despite being a symmetric model unable to capture the leverage effect. Asymmetric models such as GJR-GARCH (Glosten *et al.* (1993)[5]) and EGARCH (Nelson (1991)[8]) that belong to the GARCH family models are alternative models that allow volatility clustering, as well as the leverage effect and fat tails, to be taken into account. Bouchaud *et al.* (2001)[2] have also contributed to this literature by introducing a new way to capture the leverage effect with their retarded volatility model in which the variations in assets' prices no longer depend on the last price but rather on the exponential average of the past values. Their model works well in practice but it has one particular drawback: it only captures the retarded effect for individual stocks but does not take into account the panic effect for indices. To overcome this problem, Valeyre *et al.* (2013)[9] have recently proposed a new way to take into account both the retarded effect (specific risk) and the panic effect (systematic effect) when modelling the conditional volatility by introducing the reactive model, which is built out of an Exponentially Weighted Moving Average volatility model (EWMA)[1] based on filtered homoscedastic returns.

Even if these univariate volatility models allow us to capture the empirical properties[2] of asset returns, the problem remains in multivariate framework partly due to the so-called

---

[1]Note that an EWMA model may be seen as a degenerate GARCH(1,1) model.

[2]A complete overview of the stylized facts of asset returns is given in Mandelbrot (1967)[6] and Cont (2001)[4].

"curse of dimensionality" problem that induces an explosion of the number of parameters and then generates large uncertainties concerning the model estimates and the forecasting performances. In this paper, we aim to introduce a new covariance model, easy to implement, which allows us to capture the main stylized facts of asset returns when modelling their joint distribution. Our new model is called the reactive covariance model and comes from the generalization to the multivariate framework of the reactive volatility model introduced by Valeyre *et al.* (2013). Later in this paper, we provide the theoretical formulas before examining the properties of this new covariance estimator. We then present its main attractive features and explore its ability to capture the dominant factors that create changes in the correlation structure of asset returns. By comparing our model with other standard existing covariance models, we finally discuss and present the advantages of using this new model in asset allocation.

The paper is structured as follows. In Section 2, we briefly review the construction of the reactive volatility model. Section 3 extends this particular model to the multivariate framework by both introducing the reactive covariance model and presenting its main mathematical and empirical properties. Section 4 outlines the methodology used to assess the value of this new covariance estimator and presents the empirical results while section 5 provides the main conclusions.

2. The reactive volatility model

In order to capture the main stylized facts of asset returns such as the volatility clustering (large - respectively small price changes are followed by other large - respectively small changes), the leverage effect (negative returns tend to cause greater increases in volatility than the one induced by positive returns of the same magnitude) and the conditional heteroskedasticity (the conditional volatility is not constant), Valeyre *et al.* (2013) have recently proposed a new volatility model called the reactive volatility model. In this section, we give a brief review and the key concepts of the construction of this model before extending it to the multivariate framework.

Let I_t denote an index price at discrete time t. As empirical arithmetic returns defined

by $\frac{\Delta I_t}{I_t} := \frac{I_{t+1} - I_t}{I_t}$ are heteroscedastic due to the price-volatility correlations, Valeyre *et al.* (2013) have defined a level L_t for an index price from two exponentially moving average levels denoted L^s and L^f (which represent respectively the slow decay for specific risk and fast decay for systematic risk) in order to construct the homoscedastic returns that they have defined by $\frac{\Delta I_t}{L_t}$. These two levels are formally defined as below:

$$L_{t+1}^s = (1 - \lambda_s)L_t^s + \lambda_s I_{t+1} \tag{2.1}$$

$$L_{t+1}^f = (1 - \lambda_f)L_t^f + \lambda_f I_{t+1} \tag{2.2}$$

where $\lambda_s = 0.0241$ and $\lambda_f = 0.1484$ represent[3] respectively the inverse of the relaxation time for the retarded or specific risk effect and the panic or systematic risk effect (see Bouchaud *et al.* (2001) for more details). In order to make L^s more robust to extreme events, the authors have introduced a filter F_ϕ in order to build a new level $\hat{L}^s$ defined as:

$$\hat{L}_{t+1}^s = I_{t+1}\big(1 + F_\phi\big(\frac{L_{t+1}^s - I_{t+1}}{I_{t+1}}\big)\big) \tag{2.3}$$

where the filter function was chosen as $F_\phi(z) = \frac{tan(\phi z)}{\phi}$ with[4] $\phi = 3.3$. The level L used to construct the homoscedastic returns is then defined from both levels L^s and L^f by writing:

$$L_{t+1} = \hat{L}_{t+1}^s\left(1 + F_\phi\left(\left(\frac{L_{t+1}^f}{I_{t+1}}\right)^l - 1\right)\right) \tag{2.4}$$

where l represents a leverage parameter set[5] at $l = 8$ (the authors found and argued that this value parameter is universal as it is stable over time and is approximately the same for different mature of stock markets). Using the first order Taylor expansion, equation (2.4) can be derived as:

$$L_{t+1} \approx \hat{L}_{t+1}^s\left(1 + l\left(\frac{L_{t+1}^f - I_{t+1}}{L_{t+1}^f}\right)\right) \tag{2.5}$$

---

[3]These values were calibrated from the leverage correlation function defined in Bouchaud *et al.* (2001).

[4]They have set $\phi = 1/0.3 \approx 3.3$, which corresponds to a maximum stock index daily variation of $\pm 30\%$ or a maximum drawdown on the order of 30% over 40 days ($1/\lambda_s \approx 40$).

[5]The value of $l = 8$ of the leverage parameter means that if the index varies by 1%, the volatility is expected to vary by $-l * 1\% = -8\%$.

This level L is then used to calculate an exponential moving average volatility process $\tilde{\sigma}_I$ based on re-normalized returns $\frac{\Delta I_t}{L_t}$:

$$(\tilde{\sigma}_{t+1}^I)^2 = (1 - \lambda_\sigma)(\tilde{\sigma}_t^I)^2 + \lambda_\sigma \left(\frac{\Delta I_t}{L_t}\right)^2 \tag{2.6}$$

where the weighting parameter[6] $\lambda_\sigma = 0.025$. They found that this estimator built on homoscedastic returns $\tilde{\sigma}_I$ is both stable and able to capture the volatility variations in the long term. The reactive volatility estimator for an index σ_I is then obtained by modulating $\tilde{\sigma}_I$ with the factor $\frac{L_{t+1}}{I_{t+1}}$ which helps to capture the instantaneous price changes:

$$\sigma_{t+1}^I = \tilde{\sigma}_{t+1}^I \frac{L_{t+1}}{I_{t+1}} \approx \tilde{\sigma}_{t+1}^I \frac{L_{t+1}^s}{I_{t+1}} \left(1 + l\left(\frac{L_{t+1}^f - I_{t+1}}{L_{t+1}^f}\right)\right) \tag{2.7}$$

When dealing with single stocks i with prices P_t^i, the equivalent reactive volatility estimator is obtained by replacing the index price I_t with the stock price P_t^i in equations (2.1) and (2.3) while keeping the systematic risk relative to an index I in equation (2.2) unchanged. The reactive volatility estimator formula for a single stock is then defined as:

$$\sigma_{t+1}^i = \tilde{\sigma}_{t+1}^i \frac{L_{i,t+1}}{P_{i,t+1}} \approx \tilde{\sigma}_{t+1}^i \frac{L_{i,s,t+1}^s}{P_{i,t+1}} \left(1 + l\left(\frac{L_{t+1}^f - I_{t+1}}{L_{t+1}^f}\right)\right) \tag{2.8}$$

Looking at the second part of equation (2.8), we can see one of the key strengths of the reactive volatility formula for single stocks: its ability to take into account both the specific risk (retarded effect) and the systematic risk (panic effect). With this interesting feature, one can expect to capture in a reactive way both the idiosyncratic risk and the systematic risk of single stocks that make up an equity portfolio or a diversified portfolio. Figure 1 gives an example of the calibration of the reactive volatility model on an equity index and a bond index. We can observe that the slow level captures the long term movements of the index price changes while the fast level captures its instantaneous variations. While the re-normalized volatility estimator (from equation (2.6)) appears to be less volatile in short term than the reactive volatility, it is still able to capture the long-term moves.

---

[6] They have set $\lambda_\sigma = 1/40 = 0.025$ since economic uncertainty does not change significantly in a period of 2 months (40 trading days).
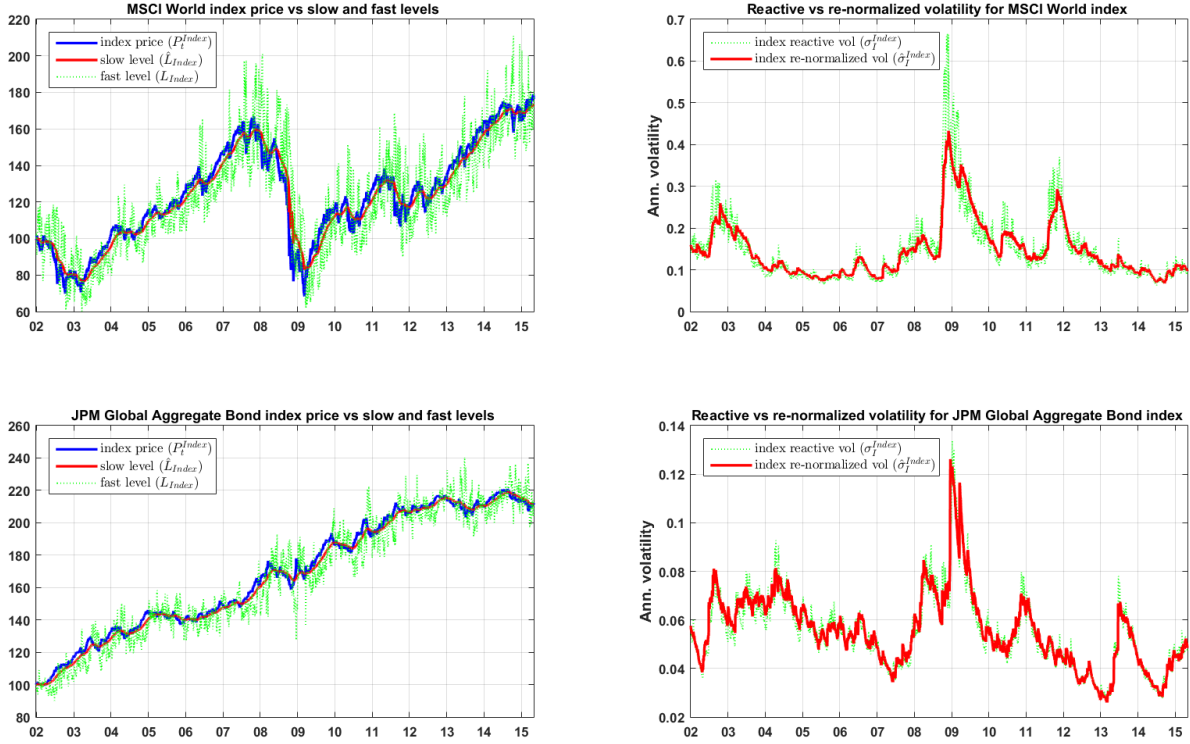
FIG 1. *Level prices and reactive volatility estimates for the MSCI World and JPM Global Aggregate Bond indices.* *Upper and lower left: The slow level (red) captures the long term movements of the index price changes (blue) while the fast level (green) captures the instantaneous variations. Upper and lower right: the re-normalized volatility (red) is more stable than the reactive volatility (green).*

3. The reactive covariance model

Let us start by highlighting the volatility limits and the importance of extending our study to the multivariate framework. In fact, the volatility is unable to capture higher dimensional information as it is possible to have two portfolios with the same volatility but with different intrinsic characteristics. The volatility therefore conceals some higher dimensional information which is important to estimate when building hedging strategies or designing portfolios for instance. In the following section, we extend the reactive volatility model presented above to the multivariate framework and present its theoretical validation.

3.1. Definition

In order to write the reactive covariance model, we have built a reactive covariance operator inspired by the construction of the reactive volatility model. For each pair of stocks $(i, j)_{i \leq j}$, the reactive covariance operator is then built out of an EMA covariance on homoscedastic returns.

Definition 3.1 (Reactive covariance) *The reactive covariance between two assets (i, j) is defined by the following equations :*

$$\widetilde{\Sigma}_{t+1}^{ij} = (1 - \lambda_\sigma)\widetilde{\Sigma}_t^{ij} + \lambda_\sigma \left(\frac{\Delta P_t^i}{L_t^i} \frac{\Delta P_t^j}{L_t^j} \right) \tag{3.1}$$

$$\Sigma_{t+1}^{ij} = \widetilde{\Sigma}_{t+1}^{ij} \frac{L_{t+1}^i}{P_{t+1}^j} \frac{L_{t+1}^j}{P_{t+1}^j} \tag{3.2}$$

Remark 3.1 *Recall that for a single stock, the reactive volatility was calculated by taking an EMA on re-normalized homoscedastic returns $\frac{\Delta P_t^i}{L_t^i} = \frac{P_{t+1}^i - P_t^i}{L_t^i}$. Taking $i = j$ in equations (3.1) and (3.2) yields the reactive volatility of Section 2 :*

$$(\widetilde{\sigma}_{t+1}^i)^2 = (1 - \lambda_\sigma)(\widetilde{\sigma}_t^i)^2 + \lambda_\sigma \left(\frac{\Delta P_t^i}{L_t^i} \right)^2 \tag{3.3}$$

$$\sigma_{t+1}^i = \widetilde{\sigma}_{t+1}^i \frac{L_{t+1}^i}{P_{t+1}^i} \tag{3.4}$$

Remark 3.2 *Reiterating equation (3.1), we get :*

$$\widetilde{\Sigma}_t^{ij} = \lambda_\sigma \sum_{k=1}^{\infty} (1 - \lambda_\sigma)^{k-1} \frac{\Delta P_{t-k}^i}{L_{t-k}^i} \frac{\Delta P_{t-k}^j}{L_{t-k}^j} \tag{3.5}$$

We get another expression for the reactive covariance[7] :

$$\Sigma_t^{ij} = \left(\lambda_\sigma \sum_{k=1}^{\infty} (1 - \lambda_\sigma)^{k-1} \frac{\Delta P_{t-k}^i}{L_{t-k}^i} \frac{\Delta P_{t-k}^j}{L_{t-k}^j} \right) \frac{L_t^i}{P_t^i} \frac{L_t^j}{P_t^j} \tag{3.6}$$

Equation (3.5) corresponds simply to an EWMA covariance estimator computed on homoscedastic returns. It clearly assumes a linear relationship between homoscedastic returns of two assets and is able to capture the long term variations. Re-normalizing this expression by the reactive factor $\frac{L_t^i}{P_t^i} \frac{L_t^j}{P_t^j}$ yields clearly to a non-linear estimator as shown in

---

[7]Similar to the univariate case, we use $\lambda_\sigma = 0.025$.

equation (3.6). The reactive covariance estimator is therefore able to account for the long term linear relationship between two assets while incorporating non-linear effects in instantaneous price changes. However, from a mathematical perspective, the re-normalization leads to a non-bilinear operator which could be problematic when computing the reactive volatility of a portfolio from the reactive covariance matrix of its constituents. This highlights once again the tradeoff that one has to deal with when modelling the covariance matrix between model complexity and mathematical tractability. The aim of the next section is to make some theoretical approximations and test them empirically in order to reconcile the two perspectives.

3.2. Theoretical validation

We start by checking if the operator of equation (3.6) is well defined from a theoretical point of view. Indeed, a covariance operator should define a dot product $\langle .,. \rangle$ on $\mathbb{R}^n$. As a reminder, $\langle .,. \rangle$ is a dot product if and only if it is positive semi-definite ($\forall x, \langle x, x \rangle \geq 0$), symmetric ($\forall x, y, \langle x, y \rangle = \langle y, x \rangle$) and bilinear ($\forall x, y, z \in \mathbb{R}^n, \forall \alpha, \beta \in \mathbb{R}, \langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$). It must verify the Cauchy-Schwarz inequality: $\langle x, y \rangle \leq \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle}$, which ensures that the correlation is well defined in $[-1, 1]$. In our case, we can directly apply this inequality to easily show that $\Sigma^{ij} \leq \sigma^i \sigma^j$ since the operator is defined as a sum. This allows us to define the reactive correlation and ensures that it takes values in the interval $[-1, 1]$.

Proposition 3.1 *The reactive correlation $\rho^{ij} := \frac{\Sigma^{ij}}{\sigma^i \sigma^j} \in [-1, 1]$ is well defined $\forall 1 \leq i, j \leq n$.*

PROOF. Using the triangle and the Cauchy-Schwarz inequalites, we have:

$$
\begin{aligned}
\left| \Sigma_t^{ij} \right| &= \left| \left(\lambda_\sigma \sum_{k=1}^\infty (1 - \lambda_\sigma)^{k-1} \frac{\Delta P_{t-k}^i}{L_{t-k}^i} \frac{\Delta P_{t-k}^j}{L_{t-k}^j} \right) \frac{L_t^i}{P_t^i} \frac{L_t^j}{P_t^j} \right| \\
&\leq \left(\sum_{k=1}^\infty \left(\sqrt{\lambda_\sigma (1 - \lambda_\sigma)^{k-1}} \left| \frac{\Delta P_{t-k}^i}{L_{t-k}^i} \frac{L_t^i}{P_t^i} \right| \right) \left(\sqrt{\lambda_\sigma (1 - \lambda_\sigma)^{k-1}} \left| \frac{\Delta P_{t-k}^j}{L_{t-k}^j} \frac{L_t^j}{P_t^j} \right| \right) \right) \\
&\leq \left(\sum_{k=1}^\infty \lambda_\sigma (1 - \lambda_\sigma)^{k-1} \left(\frac{\Delta P_{t-k}^i}{L_{t-k}^i} \frac{L_t^i}{P_t^i} \right)^2 \right)^{\frac{1}{2}} \left(\sum_{l=1}^\infty \lambda_\sigma (1 - \lambda_\sigma)^{l-1} \left(\frac{\Delta P_{t-l}^j}{L_{t-l}^j} \frac{L_t^j}{P_t^j} \right)^2 \right)^{\frac{1}{2}} \\
&= \sigma_t^i \sigma_t^j
\end{aligned}
$$

10

□

We easily verify that the reactive covariance operator is a symmetric and positive semi-definite matrix. But what about the bi-linearity? In other words, could we write the reactive variance of a portfolio on a pool of stocks as the double weighted sum of their paired reactive covariance? (*i.e.* is $(\sigma_t^{portfolio})^2 = \sum\limits_{1 \leq i,j \leq n} \alpha_i \alpha_j \Sigma_{reactive}^{ij}$?)

Proposition 3.2 *The reactive covariance operator is a symmetric and positive semi-definite matrix. Moreover, if $\forall k > 0$, $\frac{P_{t-k}^i}{L_{t-k}^i} = \frac{P_t^i}{L_t^i}$, then this operator is bilinear.*

PROOF. Let $(P_t^i)_{1 \leq i \leq n}$ denote the historical asset prices at date t. Let define by $R_{t+1}^i = \frac{\Delta P_t^i}{P_t^i} = \frac{P_{t+1}^i - P_t^i}{P_t^i}$ the heteroscedastic returns and by $\widetilde{R}_{t+1}^i = \frac{\Delta P_t^i}{L_t^i}$ the homoscedastic returns $(i = 1, ...n,\ t = 1, ..., T)$. We have therefore the relation:

$$R_t^i = \widetilde{R}_t^i \frac{L_t^i}{P_t^i} \tag{3.7}$$

Let Q be a portfolio built on these n assets and $(\alpha_1, ..., \alpha_n)$ be the vector of weights. We can write the heteroscedastic returns of the portfolio R_t^Q using the heteroscedastic returns of the assets $(R_t^i)_{1 \leq i \leq n}$:

$$R_t^Q = \sum_{i=1}^n \alpha_i R_t^i \tag{3.8}$$

We then obtain the homoscedastic return $\widetilde{R}_Q(t)$ using equation (3.7):

$$\widetilde{R}_t^Q = \frac{P_t^Q}{L_t^Q} R_t^Q = \frac{P_t^Q}{L_t^Q} \sum_{i=1}^n \alpha_i R_t^i = \frac{P_t^Q}{L_t^Q} \sum_{i=1}^n \alpha_i \frac{L_t^i}{P_t^i} \widetilde{R}_t^i \tag{3.9}$$

For the reactive covariance of the portfolio Q, we get:

11

$$
\begin{aligned}
(\sigma_t^Q)^2 &= \left(\frac{L_t^Q}{P_t^Q}\right)^2 \left(\lambda_\sigma \sum_{k=1}^{\infty}(1-\lambda_\sigma)^{k-1}\left(\frac{\Delta P_{t-k}^Q}{L_{t-k}^Q}\right)^2\right) \\
&= \left(\frac{L_t^Q}{P_t^Q}\right)^2 \left(\lambda_\sigma \sum_{k=1}^{\infty}(1-\lambda_\sigma)^{k-1}\left(\widetilde{R}_{t-k}^Q\right)^2\right) \\
&= \left(\frac{L_t^Q}{P_t^Q}\right)^2 \left(\lambda_\sigma \sum_{k=1}^{\infty}(1-\lambda_\sigma)^{k-1}\left(\frac{P_{t-k}^Q}{L_{t-k}^Q}\sum_{i=1}^{n}\alpha_i \frac{L_{t-k}^i}{P_{t-k}^i}\widetilde{R}_{t-k}^i\right)^2\right) \\
&= \left(\frac{L_t^Q}{P_t^Q}\right)^2 \left(\lambda_\sigma \sum_{k=1}^{\infty}(1-\lambda_\sigma)^{k-1}\sum_{1\le i,j\le n}\alpha_i\alpha_j \frac{L_{t-k}^i}{P_{t-k}^i}\frac{L_{t-k}^j}{P_{t-k}^j}\widetilde{R}_{t-k}^i\widetilde{R}_{t-k}^j\left(\frac{P_{t-k}^Q}{L_{t-k}^Q}\right)^2\right) \\
&= \left(\frac{L_t^Q}{P_t^Q}\right)^2 \left(\sum_{1\le i,j\le n}\alpha_i\alpha_j\lambda_\sigma \sum_{k=1}^{\infty}(1-\lambda_\sigma)^{k-1}\frac{L_{t-k}^i}{P_{t-k}^i}\frac{L_{t-k}^j}{P_{t-k}^j}\widetilde{R}_{t-k}^i\widetilde{R}_{t-k}^j\left(\frac{P_{t-k}^Q}{L_{t-k}^Q}\right)^2\right)
\end{aligned}
$$

Under the following assumption : $\forall k > 0$, $\dfrac{P_{t-k}^Q}{L_{t-k}^Q} = \dfrac{P_t^Q}{L_t^Q}$ and $\dfrac{L_{t-k}^i}{P_{t-k}^i} = \dfrac{L_t^i}{P_t^i}$, we get :

$$
\begin{aligned}
(\sigma_t^Q)^2 &= \left(\frac{L_t^Q}{P_t^Q}\right)^2 \left(\sum_{1\le i,j\le n}\alpha_i\alpha_j\lambda_\sigma \sum_{k=1}^{\infty}(1-\lambda_\sigma)^{k-1}\frac{L_{t-k}^i}{P_{t-k}^i}\frac{L_{t-k}^j}{P_{t-k}^j}\widetilde{R}_{t-k}^i\widetilde{R}_{t-k}^j\left(\frac{P_{t-k}^Q}{L_{t-k}^Q}\right)^2\right) \\
&= \left(\frac{L_t^Q}{P_t^Q}\right)^2 \left(\sum_{1\le i,j\le n}\alpha_i\alpha_j\lambda_\sigma \sum_{k=1}^{\infty}(1-\lambda_\sigma)^{k-1}\frac{L_t^i}{P_t^i}\frac{L_t^j}{P_t^j}\widetilde{R}_{t-k}^i\widetilde{R}_{t-k}^j\left(\frac{P_t^Q}{L_t^Q}\right)^2\right) \\
&= \left(\frac{L_t^Q}{P_t^Q}\right)^2 \left(\frac{P_t^Q}{L_t^Q}\right)^2 \left(\sum_{1\le i,j\le n}\alpha_i\alpha_j\left(\lambda_\sigma \sum_{k=1}^{\infty}(1-\lambda_\sigma)^{k-1}\widetilde{R}_{t-k}^i\widetilde{R}_{t-k}^j\right)\frac{L_t^i}{P_t^i}\frac{L_t^j}{P_t^j}\right) \\
&= \sum_{1\le i,j\le n}\alpha_i\alpha_j\Sigma^{ij} \qquad\qquad (3.10)
\end{aligned}
$$

Under this assumption the reactive covariance operator defined by equation (3.6) is bilinear. $\qquad\square$

We verify empirically that this assumption is not very restrictive by running a procedure that consists of using daily data of a pool of four equity indices and four bond indices between January 2003 and May 2015. We compute at each date time t the reactive volatility of the equally-weighted portfolio based on these data using two different approaches:

1. In the first approach, we calibrate the reactive volatility estimates $\hat{\sigma}_t^{react}$ of the equally-weighted portfolio using equations (3.3) and (3.4) directly on the time series of its empirical returns.

2. In the second approach, we estimate the reactive covariance matrix $\hat{\Sigma}_t^{react}$ using equations (3.1) and (3.2) and compute the following quantity[8]: $\frac{1}{N}\sqrt{e'\hat{\Sigma}_t^{react}e}$.
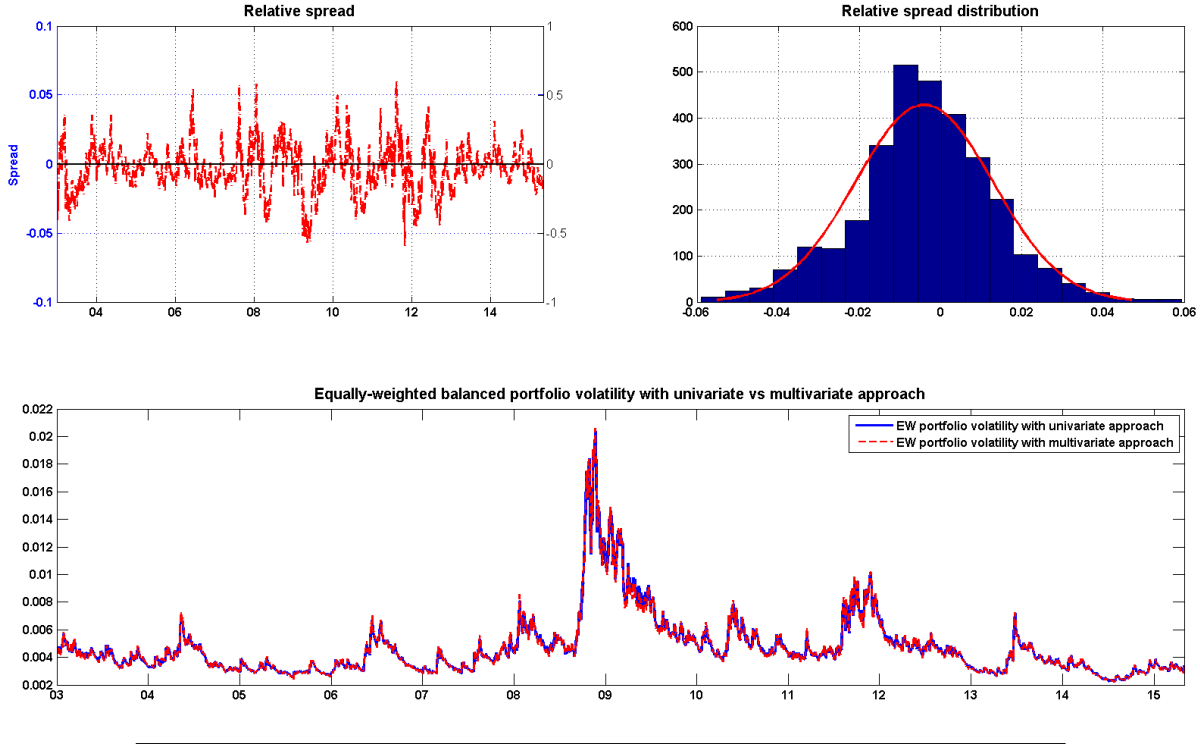

FIG 2. **Relative spread** $\left(\left(\frac{1}{N}\sqrt{e'\hat{\Sigma}_t}e - \hat{\sigma}_t\right)/\hat{\sigma}_t\right)_t$. *Upper left and right: the evolution of the relative spread is bounded by 5% and the distribution of the relative spread (blue) with a fitted gaussian (red) is concentrated around 0. Lower: The equally-weighted portfolio reactive volatility with univariate (blue) vs multivariate approach (dotted red).*

If the assumption is not very restrictive, we should get the quasi-bilinearity of the reactive covariance estimator, *i.e.* $\hat{\sigma}_t^{react} \approx \frac{1}{N}\sqrt{e'\hat{\Sigma}_t^{react}e}$. Figure 2 shows the distribution and the evolution of the relative spread $(\frac{1}{N}\sqrt{e'\hat{\Sigma}_t}e - \hat{\sigma}_t)/\hat{\sigma}_t$ over time. It follows that the assumption is not too restrictive since the distribution is concentrated around 0. Moreover, in the worst case, the relative spread is bounded by 5%.

---

[8]Where e stands for the $N \times 1$ vector of ones.

4. The reactive covariance model: implications in asset allocation

4.1. Data description

To make the empirical tests, we have considered a balanced portfolio that is meant to represent a well diversified investment with both exposure to bond and equity risk. The sample comes from Bloomberg and consists of a dataset of eight asset classes (four bond and four equity indices) taken at daily frequency in total return term and denominated in US Dollar. The bond group contains two government bond indices and two corporate bond indices represented by the 10-year US bond (Barclays US Govt 10Y), the 10-year European bond (Barclays EMU Govt 10Y), the US Investment Grade Bonds (Barclays US Corp IG TR) and the US High Yield Bonds (Barclays US Corp HY TR). The equity group is made of four regional equity indices that represent US equities (MSCI USA TR), EURO equities (MSCI EMU TR USD), Japanese equities (MSCI Japan TR USD) and Emerging Market equities (MSCI EM TR USD). The sample spans the period January 2, 2002 to May 1, 2015 and the LIBOR USD 1 Month index is used to calculate the excess returns and Sharpe ratios.

Descriptive statistics over the entire sample are given in Table 1. We clearly observe that there is a large heterogeneity in terms of individual returns, volatility and correlation coefficients when considering this balanced portfolio. In the following section, we will check if the reactive covariance model helps to manage this heterogeneity properly, especially in times of market turbulence.

TABLE 1

Historical descriptive statistics for the balanced portfolio (Jan 2002 – May 2015)

Asset classes	Ann. Returns (%)	Ann. Vol (%)	Correlations (%)							
US Bonds 10Y	6.10	7.00	100	57.49	90.77	-7.44	-38.11	-32.24	4.93	-24.46
EU Bonds 10Y	6.77	4.74		100	51.26	-11.09	-29.06	-38.14	2.35	-24.90
US IG	5.96	4.99			100	18.81	-29.01	-17.23	15.26	-6.31
US HY	8.65	5.01				100	26.31	43.78	29.81	51.98
US Equities	8.63	20.08					100	58.97	5.03	46.24
EU Equities	9.49	25.58						100	23.88	67.59
Japan Equities	7.98	22.60							100	48.47
EM Equities	13.90	20.56								100

4.2. Methodology

We explore the values of the reactive covariance model within a *risk parity* or *risk budgeting* framework. The risk parity approach provides a risk-based investment style that doesn't rely on return forecasts and where the risk contribution of each component of the portfolio is equal to a certain risk budget defined by the portfolio manager. This new framework is considered today as the main alternative method to the traditional mean-variance portfolio optimization and its theoretical properties can be found in Bruder and Roncalli (2012) [3].

To broadly present the risk parity approach, let us consider a portfolio of n assets. We define by x_i the weight of the ith asset and by $\Re(x_1, ..., x_n)$ the risk measure of the portfolio $x = (x_1, ..., x_n)$. In this paper, we focus on the volatility as the risk measure and therefore consider the case $\Re(x) = \sigma(x) = \sqrt{x'\Sigma x}$ where Σ denotes the variance covariance matrix (VCV) of the portfolio. Considering this case, the risk contribution of an asset i is then defined by:

$$\Re C_i(x_1, ..., x_n) = x_i \frac{\partial \Re(x_1, ..., x_n)}{\partial x_i} = x_i . \frac{(\Sigma x)_i}{\sqrt{x'\Sigma x}} \tag{4.1}$$

As the risk budgeting approach consists in building the portfolio such that the risk contribution matches a given risk budget $b_i = (b_1, ..., b_n)$, our risk parity portfolio is then defined by the following constraint:

$$\Re C_i(x_1, ..., x_n) = b_i * \Re(x_1, ..., x_n) \tag{4.2}$$

The general mathematical system that defines our optimization problem is therefore:

$$\begin{cases} x_i.(\Sigma x)_i = b_i.(x'\Sigma x), \\ \quad b_i > 0, \quad x_i > 0 \\ \displaystyle\sum_{i=1}^{n} b_i = 1, \quad \sum_{i=1}^{n} x_i = 1 \end{cases} \tag{4.3}$$

For empirical tests, we have considered two cases to define the vector of budget b_i. In the first case, we have considered $b_i = \dfrac{1}{n}$ which corresponds to the well-known equally-weighted risk contribution case (ERC) case introduced by Maillard *et al.* (2010) [7] while in the second case, we have considered a risk budgeting (RB) process with $b_i = \dfrac{\sigma_i^{-1}}{\sum_{k=1}^{n} \sigma_k^{-1}}$.

The intuition behind the second case is that we want to allocate the portfolio's weights according to the inverse of their volatility level in order to check if the reactive covariance model provides accurate estimates of asset volatilities that could help to reduce the portfolio risk especially in times of market shock and violent rally.

To take into account the effect of equity and bond systematic risk when estimating the single stocks reactive volatilities and covariances of our multi-asset portfolio, we use the MSCI World index and JPM Global Aggregate Bond index as proxies of the equity and bond systematic risks in order to estimate the level L (as defined in equation (2.5)) and also consider the equally-weighted portfolio made of these two indices as the benchmark of our balanced portfolio. We compare the risk parity portfolios obtained from reactive covariance estimates with two standard existing models, namely the empirical covariance matrix and the EWMA covariance matrix (as defined by Risk-Metrics). We estimate the three VCVs in a one year rolling window time frame, calculate the risk parity weights at a daily re-balancing frequency and provide the out-of-sample optimal portfolios over the period January 2003 to May 2015.

4.3. Empirical results

The summary statistics of optimal ERC and RB portfolios are presented in Table 2. Compared to the classical historical covariance matrix or to the EWMA covariance matrix, we observe that the portfolios generated with the reactive covariance model exhibit higher returns with lower (versus empirical portfolio) or equal (versus with EWMA portfolios) risk. Higher Sharpe ratios are not generated at the expense of a higher drawdown that stay lower for reactive portfolios, mainly with ERC portfolios. In a relative world, the advantages of portfolios with reactive covariance matrices persist even if excess returns are generated with slightly higher tracking error. Over the whole period, excess returns are significantly higher for reactive covariance portfolios compared to empirical and EWMA models whatever the risk parity approaches considered.

The cumulative ERC and RB portfolios' performances over the out-of-sample period are given in Figure 3. This graph highlights the ability of the reactive covariance matrices to generate portfolios with better risk/return/drawdown profiles in times of market turbulence compared to empirical or EWMA covariance matrices. This interesting feature of

16

TABLE 2
Risk Parity optimal portfolios statistics using different covariance matrices (Jan 2003 - May 2015)

Statistics	Benchmark 50% Equity + 50% Bond	ERC Portfolios			RB Portfolios		
		Empirical VCV	EWMA VCV	Reactive VCV	Emp VCV	EWMA VCV	Reactive VCV
Ann. arthm mean	6.38%	6.20%	6.58%	7.00%	5.82%	6.49%	7.27%
Ann. geo mean	6.15%	6.30%	6.71%	7.15%	5.92%	6.63%	7.47%
Ann. vol	9.09%	4.30%	4.23%	4.20%	3.71%	3.66%	3.69%
Max drawdown	-33.79%	-14.76%	-13.46%	-11.71%	-16.22%	-11.72%	-9.91%
Sharpe ratio	0.70	1.44	1.56	1.67	1.57	1.77	1.97
Excess return cumulated	–	3.71%	14.02%	25.68%	-5.54%	12.11%	34.44%
Ann. excess return	–	0.15%	0.56%	1.00%	-0.23%	0.49%	1.32%
Ann. realized tracking error	–	7.29%	7.99%	8.07%	8.06%	8.73%	8.73%
Information ratio	–	0.02	0.07	0.12	-0.03	0.06	0.15
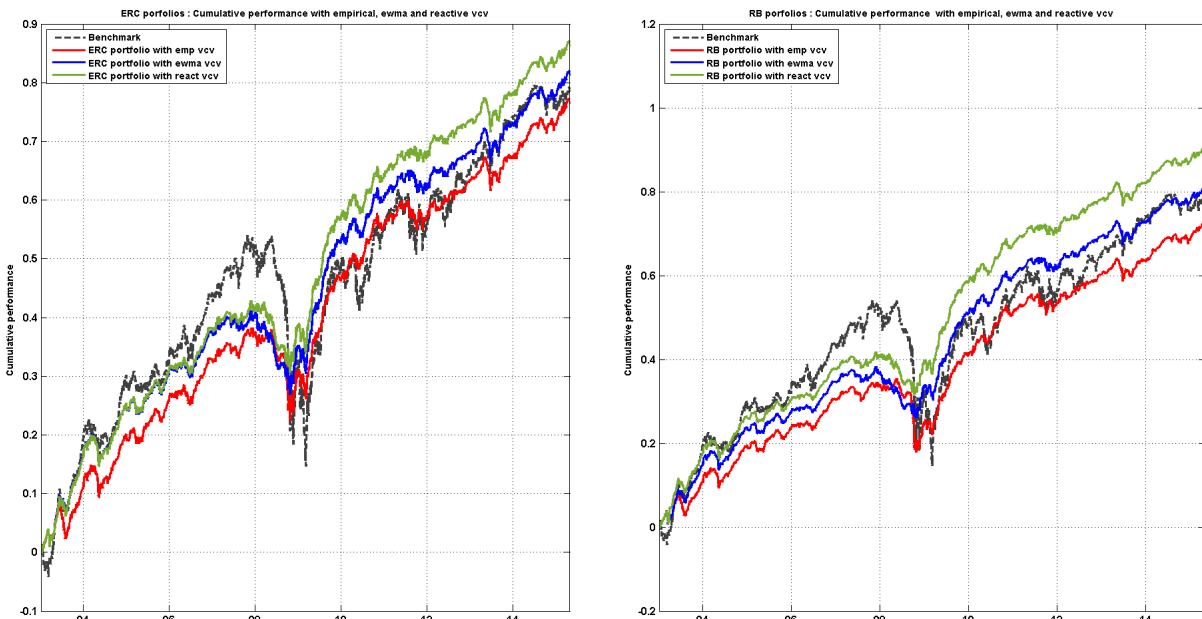
FIG 3. **Cumulative portfolio performances** of asset classes that made up the balanced portfolio over the period January 2003 to May 2015. Left: the cumulative performance of ERC portfolios. Right: the cumulative performance of RB portfolios (risk budget with volatility weighted). Portfolios are obtained using empirical, EWMA or reactive covariance matrices.

the portfolios obtained with reactive covariance matrices is emphasized in Figure 4, which provides a close-up of the cumulative portfolios performances over the period July 2007-December 2011 considered as the period where financial markets experienced different volatility regimes, bubbles and market crashes.
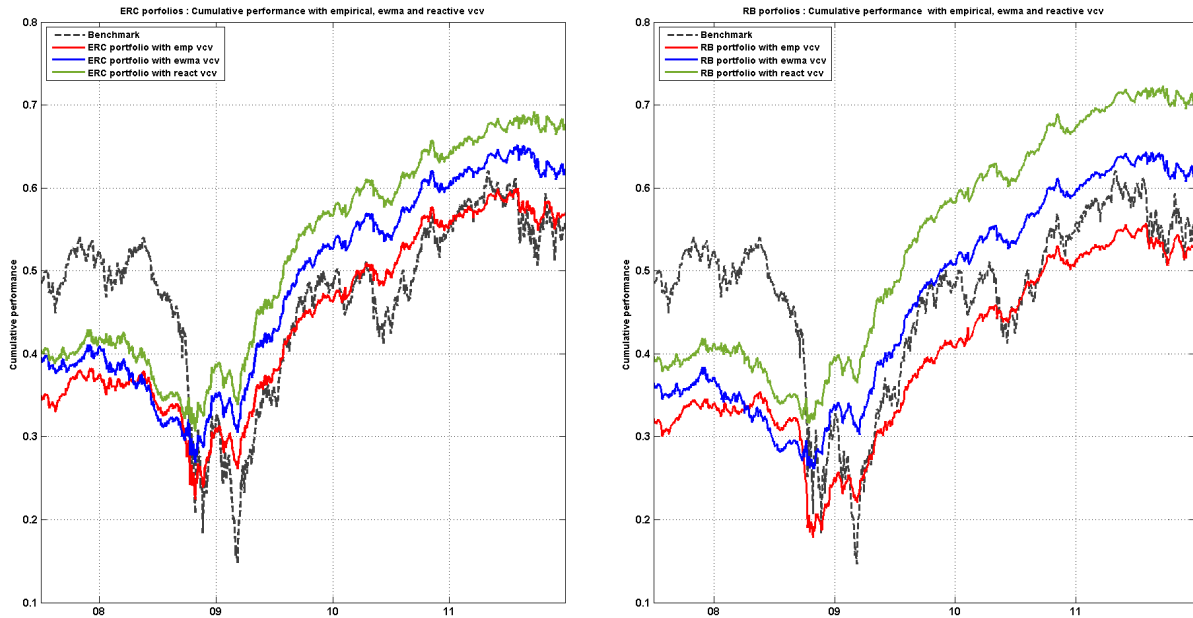
FIG 4. ***Cumulative portfolio performances*** *of asset classes that made up the balanced portfolio over the period July 2007 to December 2011. Left: the cumulative performance of ERC portfolios. Right: the cumulative performance of RB portfolios (risk budget with volatility weighted). Portfolios are obtained using empirical, EWMA or reactive covariance matrices.*

4.4. The reactive covariance: some interesting features

One of the strengths of the reactive covariance model is that it could be seen as a good proxy for systematic risk in a multi-asset framework as shown in Figure 5. This graph provides the evolution of the maximum eigenvalues of the distinct[9] reactive covariance matrices of the equity and bond groups of the balanced portfolio considered in the previous section. It clearly shows that the maximum eigenvalues of the reactive covariance matrices could be seen as a good risk factor that allows us to see when the covariance structure of a group of assets changes. For instance, when considering the risk factor of the bond group, we can see that the systemic risk factor of this group spikes in mid-2010 and 2011, in line with what happened during the recent sovereign debt crisis. The same observation can be made with the systematic risk factor from the reactive covariance maximum eigenvalues of

---

[9]We have used the whole reactive covariance matrix of the balanced portfolio to get the distinct equity and bond reactive covariance matrices and have calculated the maximum eigenvalues of the resulting matrices to obtain the plot in Figure 5.

the equity group. In times of equity market turbulence, the equity risk factor could help to identify what happens within the equity group of assets of a portfolio and therefore, could help the portfolio manager to dynamically change or reduce his equity exposure. In other words, the maximum eigenvalue of the reactive covariance matrices helps to capture more quickly the market changes of equities and bonds and therefore can be used to manage the leverage of a portfolio for example.
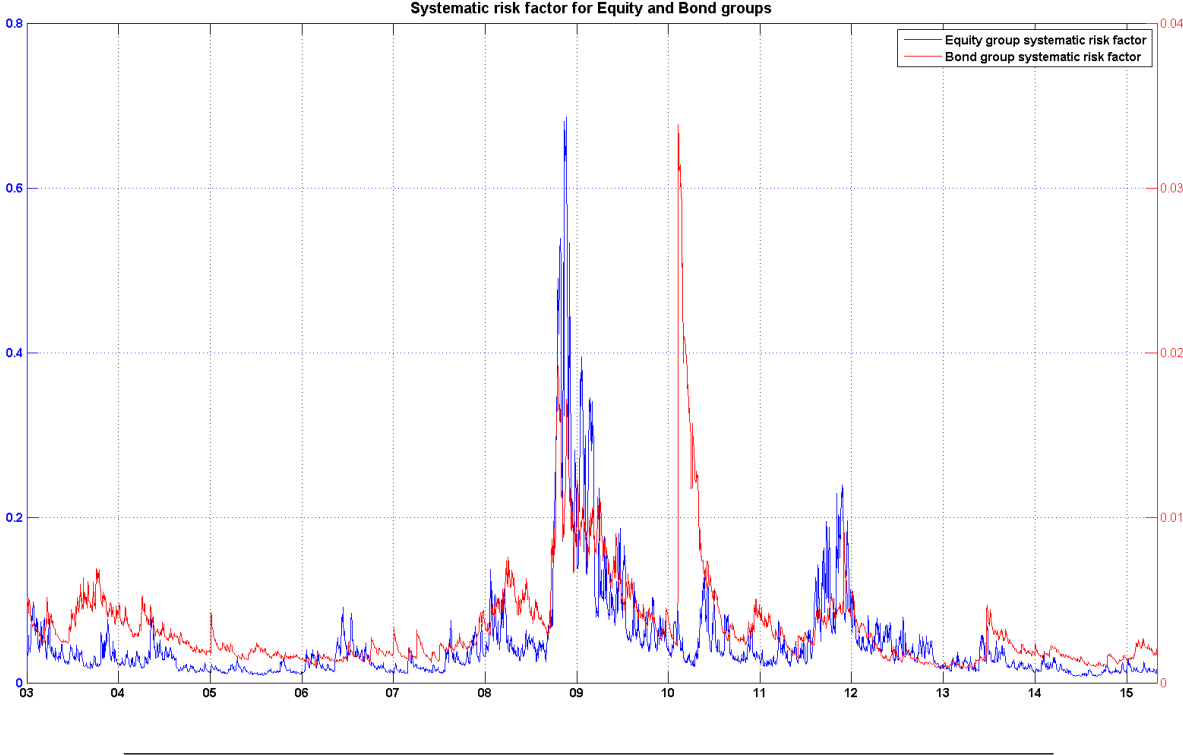

FIG 5. **Evolution of the maximum eigenvalues of reactive covariance matrices:** *a proxy for systematic risk in a multi-asset framework for the equity group (blue) and bond group (red) in a balanced portfolio.*

To highlight the fact that the systematic risk factors obtained from reactive covariance matrices could be considered as good indicators of covariance structure changes of a portfolio, we have plotted in Figure 6 the evolution of the maximum eigenvalues of the whole reactive covariance matrices of our balanced equity-bond portfolio against the VIX index. This graph also provides the 1-year rolling window correlations between the increments of risk indicators (from empirical, EWMA and reactive covariance matrices) with our equity-

19

bond portfolio benchmark as defined in section 4.2. Looking at the graphs, we can see that, like the VIX index, the systematic risk factor derived from the reactive covariance matrix is strongly negatively correlated with benchmark returns. This observation makes it an ideal candidate for strategic or tactical strategies, which is not the case for both the empirical and EWMA covariance matrix. This last observation can be seen as one of the main features of the reactive covariance matrix compared to the EWMA covariance matrix. Thus, unlike linear estimator such as the historical empirical or the EWMA covariance, the reactive covariance better reflects the systematic risk and specific risk of a pool of assets as it allows the non-linearity of the market to be captured.
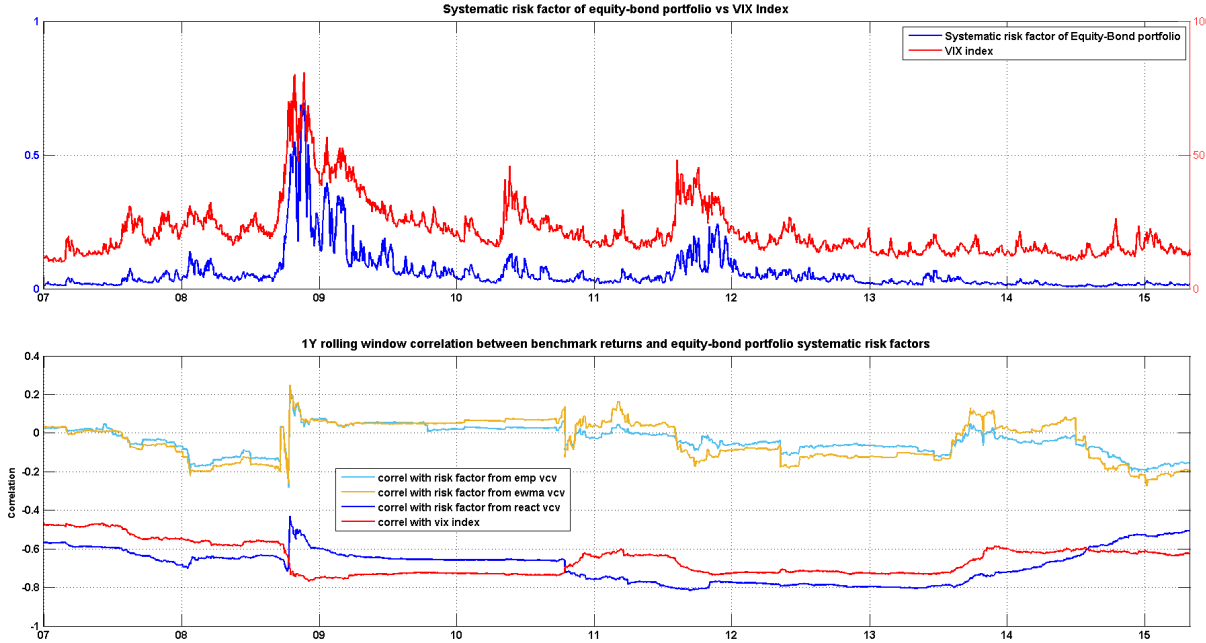
FIG 6. ***Maximum eigenvalue of reactive covariance matrix :*** *Upper: Evolution of VIX index (blue) and the maximum eigenvalues of the reactive covariance matrices of the equity-bond universe (red). Lower: One-year rolling correlations between benchmark returns and the maximum eigenvalues of the empirical covariance matrix (cyan), the EWMA matrix (yellow), the reactive covariance matrix (blue) and the VIX index (red). The systematic risk factor from reactive covariance matrices could be seen as a good proxy of systematic risk since it is strongly negatively correlated with benchmark returns: it allows to quickly capture periods of market turbulence.*

On a practical basis, the systematic risk factor based on maximum eigenvalues gives balanced portfolio managers a gauge of having a proxy of implied market equity and

bond risks without having to deal with complex or non-existent options data on both asset classes. We note that the high correlation between our systematic risk factor and the VIX is due to the historical dataset sample under study. Indeed, in the last fifteen years of financial market history, the main risk in a balanced portfolio has come from the equity class. But we could reasonably expect that over the coming years, with the US Federal Reserve engaged in raising short term rates after 8 years of unconventional monetary policy, the main risk factor may come this time from the bond market. In this scenario, the systematic risk factor derived from the reactive covariance model will bring more added value than ever to the balanced investment process. In terms of use, the new leading indicator presented in this paper (that also includes implied risk from the bond market) could be used to build market timing signals to allocate assets between low and high risk portfolios. In other words, the maximum eigenvalue risk indicator from equity-bond reactive covariance matrices could be seen as a new tool used for leverage adjustment or beta management in turbulent markets.

5. Conclusion

In this article, we have proposed a new covariance matrix estimator which comes from the generalization to the multivariate framework of the reactive volatility model. This new model allows one to model the leverage effect of single stocks of a portfolio as a function of their own idiosyncratic risk and benchmark systematic risk.

We have compared the added-value of this new estimator with standard existing ones within a risk parity framework. When applied to a multi-asset universe, our findings show that portfolios obtained with the reactive covariance model have superior diversification with minimum drawdowns and good risk-adjusted characteristics. The new proposed co-variance estimator then has the ability to quickly adapt to market changes and provides accurate estimates of the variances and covariances of a group of assets. Moreover, we also found that systematic risk factor build on the maximum eigenvalues of the reactive covariance matrices can be used as an interesting risk aversion indicator that could enable one to manage the leverage or the beta of the portfolio especially in times of market turbulence.

Further development or application of this model could be to use it to build signals that allow one to dynamically change the strategic asset allocation of a multi-asset portfolio according to market environments. Finding a multi-assets grid based on the new systematic risk factor introduced in this paper could allow the portfolios obtained with the reactive covariance model to outperform the benchmark even further. Application of the reactive covariance model in risk management when considering a single asset class framework for instance could also lead to a better managed portfolio, especially in crisis entry and exit phases, as the reactive feature of the covariance matrices could lighten the risk management constraint in a crisis exit scenario by stopping the overestimation of risk coming from an empirical measure. Finally, an interesting study could be to assess all the advantages of using the reactive covariance model in a complex portfolio optimization process with tracking error constraints.

References

[1] BOLLERSLEV, T. (1987). *A conditionally heteroskedastic time series model for speculative prices and rates of return.* The review of economics and statistics, 542-547.

[2] BOUCHAUD, J. P., & POTTERS, M. (2001). *More stylized facts of financial markets: leverage effect and downside correlations.* Physica A: Statistical Mechanics and its Applications, 299(1), 60-70.

[3] BRUDER, B., & RONCALLI, T. (2012). *Managing Risk Exposures using the Risk Budgeting Approach.* SSRN, http://ssrn.com/abstract=2009778

[4] CONT, R. (2001). *Empirical properties of asset returns: stylized facts and statistical issues.* Quantitative Finance, 1, 223-236.

[5] GLOSTEN, L. R., JAGANNATHAN, R., & RUNKLE, D. E. (1993). *On the relation between the expected value and the volatility of the nominal excess return on stocks.* The journal of finance, 48(5), 1779-1801.

[6] MANDELBROT, B. (1967).*The variation of some other speculative prices.* Journal of Business, 393-413.

[7] MAILLARD, S., RONCALLI, T., & TEÏLETCHE, J. (2010). *The properties of equally weighted risk contributions portfolios.* Journal of Portfolio Management, 36 (4), pp 60-70.

[8] NELSON, D. B. (1991). *Conditional heteroskedasticity in asset returns: A new approach.* Econometrica: Journal of the Econometric Society, 347-370.

[9] VALEYRE, S., GREBENKOV, D., ABOURA, S., & LIU, Q. (2013). *The reactive volatility model.* Quantitative Finance, 13(11), 1697-1706.

<u>Chief Editors:</u>

Pascal Blanqué
Deputy Chief Executive Officer
Head of Institutional Investors and Third Party Distributors
Group Chief Investment Officer

Philippe Ithurbide
Global Head of Research, Strategy and Analysis

Amundi Working Paper

WP-57-2016

May 2016

research-center.amundi.com