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Optimal Decumulation Strategies for Retirement Solutions

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Abstract

Traditional pension systems based on pay-as-you-go (PAYG) have become more difficult to sustain in developed countries as a result of continued population ageing, fewer children in the family and weak economic growth. Over the past decades, pension systems have gradually shifted from defined-benefit (DB) plan to defined-contribution (DC) plan. In DC pension schemes, there is no guarantee of pension payments after retirement, so it is important to develop appropriate decumulation strategies to efficiently convert wealth into income. The first part of this working paper systematically reviews two types of decumulation strategies: safe withdrawal rate methods and optimal control theory-based methods. The second part presents simulations of these strategies. In addition, we investigate the robustness of these strategies by simulating non-normally distributed returns of risky assets, such as Student's t distribution and skew normal distribution. Finally, we test these strategies with forward-looking financial data simulated by Amundi Cascade Asset Simulation Model.

Keywords: Decumulation, pension schemes, stochastic optimal control, HJB equation, Monte Carlo simulation.

JEL classification: C61, C63, E21, G11.

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1 Introduction

A pension plan is a long-term savings plan designed to protect retirees from the risk of poverty in old age by transferring a portion of their working income to retirement and is an effective means of ensuring financial stability and security after retirement. It is an important element of social security that is widely practiced in the vast majority of developed countries countries and in many developing countries around the world. A complete pension scheme consists of two parts: an accumulation phase in which people build up reserves through regular cash savings (often called contributions) during their working lives and a decumulation phase in which people receive payment to support their retirement lives.

In terms of the way in which pension schemes are financed, we can distinguish two types of pension systems:

- **Pay as you go (PAYG)**: This system implies that active workers' contributions are shared by retirees and is notable for the absence of individual bank accounts. The regular contributions paid by the workers are transferred directly to the retirees for the payment of pensions. Since contributions are not accumulated or invested in the market, this system amounts to an inter-generational contract in which the pensions paid by the current generation are used to support the previous generation of retired workers. The sustainability of such systems therefore depends on the ratio of the number of workers to the number of retirees and is particularly affected by demographic changes, such as increasing life expectancy and declining fertility.
- **Fully funded**: In this system, the contributions of active employees are deposited into individual accounts. These reserves are invested in the financial markets to obtain more value and are retained until retirement. Due to the lack of mutual assistance, this system is not subject to demographic changes but is highly vulnerable to financial risks such as market risk, inflation risk, etc. Such systems are more suitable for countries with ageing societies and highly developed and regulated financial markets.

In terms of how pension schemes are paid out, we can divide the pension system into two categories:

- **Defined-benefit (DB)**: Under this system, pension expenses and income are social pooling accounts. The pension provider (e.g., employer or government) promises retirees a determined income for life, and the amount is based on factors such as the retiree's pre-retirement salary and years of service. Defined benefit pensions can be very difficult to manage, especially in the context of an ageing population where increasing numbers of retirees lead to an imbalance between pension income and expenses. To maintain the program, the government has to make financial subsidies, which will also lead to inter-generational inequality.
- **Defined-contribution (DC)**: Under this system, each individual has his or her own account and invests in the financial market to generate income. The individual is responsible for contributing to the pension and selecting the investments offered by the pension scheme. As a result, the value of the pension can go up or down, depending on the performance of the investments. The most famous defined contribution plan is the 401(k) plan in the US.

Table 1: Comparison of DB and DC pension schemes

	Defined-benefit	**Defined-contribution**
Characteristics	The employer[1] undertakes to pay the worker a guaranteed defined benefit upon retirement.	The employer and the employee contribute a fixed percentage of salary to the employee's pension account on a regular basis to fund the pension.
Payment risk	The employer bears all the payment risk.	The employee bears all the payment risk.
Payment amount	Payments are based on a predefined formula.	Payments are dependent on the fund's investment performance.
Advantages	Guaranteed payment; No investment risk for employees	Potential for higher investment returns; Flexibility
Disadvantages	Low average retirement payments; Lack of flexibility	Uncertainty of payments; Employees have investment risks

Source: Amundi Investment Institute

Defined-benefit schemes are generally financed on a PAYG pension system or a partially fully funded system. Defined-contribution pension systems are, by definition, fully funded. Table 1 shows the main differences between DB and DC pension schemes. The sustainability of DB schemes depends on the balance between the contributions of employees and employers and the pensions drawn by retirees, i.e., the demographic structure. In the late 1970s and early 1980s, developed countries' pension systems have been affected by factors such as continued population ageing, fewer children in families, and sluggish economic growth. The statutory pensions, which were mainly financed on a PAYG basis, generally fell into the predicament of not being able to cover their expenses. Over the past few decades, there has been a gradual shift from DB to DC plans. The responsibility for preparing for retirement is shifting from governments and employers to individuals. Since the 1980s, countries have initiated reforms and chosen a similar path of reform: in addition to reducing public pensions under PAYG systems, a three-pillar pension system was formed by introducing fund accumulation, replacing the previous system of sole state funding with a system in which the state, companies, and individuals are responsible for pensions. This three-pillar system has been actively promoted by the World Bank. In their policy research report (World Bank, 1994), the World Bank describes the different objectives of each pillar.

- The first pillar aims to ensure a basic cost-of-living allowance for seniors. This would be a mandatory, publicly administered PAYG system, usually with a DB plan, funded through contributions to wages and government budgets. The first pillar is redistributive and inclusive in nature and is designed to ensure that retirees receive a pension equal to $50\% - 60\%$ of their pre-retirement salary.

[1] In this table, the term "*employer*" refers not only to companies but also to governments.

- The second pillar is a privately managed mandatory savings system designed to supplement the pension benefits of the first pillar, primarily through defined-contribution plans. It consists of occupational pension plans with mandatory enrollment.
- The third pillar refers to private retirement savings options. In contrast with the first two pillars, this pillar is voluntary. In this type of savings plan, individuals are responsible for deciding how much they will contribute and how they will invest their savings. In order to encourage individuals to save on their own, the government often offers different forms of tax advantage for certain specific savings, such as deducting contributions from taxable income, tax-free withdrawals, or partial capital gains tax.

In most countries, the second pillar usually refers to the actions of companies, which are responsible for establishing an annuity system for their employees. Therefore, in practice, the three-pillar pension model, in a narrow sense, involves public pensions, company annuities, and personal pension savings. Especially after the global financial crisis in 2008 and the sovereign debt crisis in Europe in 2010, countries have stepped up their reform efforts as a result of rising levels of public debt. Reform measures in many developed countries include:

- extension of contribution years and the statutory retirement age;
- raising the level of contributions;
- institutional reforms to reduce management costs;
- tax benefits for second and third pillar pension solutions.

At the end of 2005, the World Bank published another important report "*Old Age Income Security in the 21st Century - An International Comparison of Pension System Reforms*" (Holzmann and Hinz, 2005). The book extends the three-pillar idea and then puts forward the concept and proposal of five pillars: Non-contributory Pillar 0, which provides a minimum level of coverage to the poor; contributory Pillar 1, which is a public pension scheme on a PAYG basis to provide basic needs; mandatory or voluntary Pillar 2, which is a private occupational pension scheme to supplement Pillar 1; voluntary Pillar 3, which involves individual savings to provide for future withdrawals or annuities in various forms and the so-called Pillar 4, which establishes informal forms of coverage among family members or between generations.

In DC pension plans, where payments are not guaranteed during retirement, it is important to develop an appropriate decumulation strategy that effectively converts wealth into income. Designing such a decumulation strategy is a difficult challenge, and the control variables available to decision-makers include not only investment decisions but also withdrawal decisions. First, we need to consider a range of relevant market risk factors, such as investment risk, interest rate risk, inflation risk, and personal risk factors, such as longevity risk. Secondly, we should decide how to allocate the remaining funds to ensure a stable and sustainable income stream throughout the retirement period. Finally, we have to consider retirees' preferences, such as bequests, family circumstances, health conditions, spending habits, etc. Therefore, an appropriate decumulation strategy takes into account three aspects: sources of uncertainty, decision-making processes, and preferences. Thus, many authors claimed that the decumulation strategy is a very hard problem in finance, such as

Sharpe (2017) and Thaler (2019). In our paper, we analyze two broad classes of decumulation strategies: the safe withdrawal rate method derived from experience, such as the most popular *4%* rule, proposed by Bengen (1994, 1996, 1997) and optimal control theory-based methods, which explicitly model consumers' preferences by means of a utility function. In the second category, we first review the method of maximizing the expected discounted utility of lifetime consumption, which is also known as Merton's optimal investment-consumption problem. This problem, formulated and solved by Merton (1969), consists of deciding how much to consume and how to allocate their remaining wealth between stocks and a risk-free asset to maximize the expected discounted utility of life-time consumption. By choosing some forms of utility functions, such as the constant relative risk aversion (CARA) utility functions, and solving the associated Hamilton-Jacobi-Bellman (HJB) equation, we can find a closed-form solution where optimal consumption is proportional to the value of the investment fund. However, this approach is at odds with real-world situations in which retirees want a steady stream of payments to avoid unexpected changes, such as cutting or freezing withdrawals, which may occur when applying Merton's problem to a decumulation strategy. To address this problem, we consider in our paper another approach based on optimal control theory, from which we draw inspiration from the ruin time stochastic control problem. In our framework, we assume that all future consumption is known, i.e. decided by retirees according to their personal preferences, and then the stochastic control problem consists of deciding on the optimal allocation of the remaining funds between equities and risk-free assets in order to maximize expected utility of the ruin date of the fund. This approach is more in line with the goal of the decumulation strategy. Although we cannot find an explicit solution for the finite investment horizon case in this framework, we can still solve the HJB equation using numerical methods such as finite difference methods. We then use Monte Carlo simulations to investigate the performance of these three methods and compare them using different metrics, such as average success rate, average total consumption, etc. We also test the robustness of these approaches to non-normally distributed returns on risky assets. Finally, we test and compare the safe withdrawal rate (SWR) and the ruin date utility maximization (RDUM) methods with more realistic forward-looking financial data simulated by Amundi Cascade Asset Simulation Model (CASM).

This paper is organized as follows. In Section Two, we provide an overview of decumulation strategies and detail three common methods for decumulation strategies. In Section Three, we test different methods in the case of simulated data. In this section, we also test the robustness of these methods with non-normally distributed returns. In Section Four, we use Amundi internal model CASM to simulate forward-looking data. Finally, Section Five offers some concluding remarks.

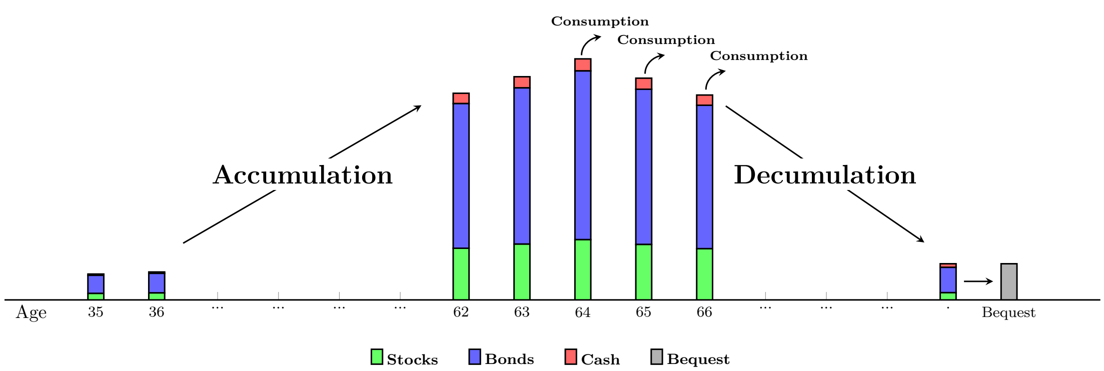
2 Decumulation strategies

Upon reaching retirement, an individual, having diligently worked and invested over the years, finds himself in possession of a certain amount of accumulated wealth, denoted as X_0. This initial wealth serves as the starting point for his or her retirement period, during which we suppose that he lacks an additional source of income. To cover their living expenses, the retiree makes frequent withdrawals from his or her fund. The remaining wealth is then allocated for investment in the financial market. Consequently, the retiree is confronted with the challenge of determining the optimal investment amount to allocate in the financial market in order to minimize the risk of outliving their financial resources. By carefully managing their investment decisions, retirees aim to strike a balance between meeting their current consumption needs and preserving their wealth to sustain their expenses throughout the retirement period.

2.1 Accumulation and decumulation

As shown in Figure 1, a complete pension scheme consists of two parts: pre-retirement accumulation and post-retirement decumulation. The accumulation phase describes the period in a person's life when savings and long-term efficient investments are made to accumulate funds for retirement. In contrast, the decumulation phase refers to the process by which investors convert their retirement savings into income withdrawals to meet their retirement needs and continue to invest their remaining funds. During this phase, there is a regular outflow of cash from the reserve. For less fortunate retirees, the appreciation of their savings invested in the financial market will not be able to meet their basic needs after retirement, and they will therefore need to spend down the principal. As a result, investment strategies during the retirement stage usually focus more on capital preservation to ensure a stable and sustainable income stream throughout the retirement period.

Figure 1: Accumulation and decumulation



Source: Amundi Investment Institute

People have different goals during periods of accumulation and decumulation. Almost all savers in the accumulation phase are focused on asset growth and have maximizing investment returns as their primary objective. However, as retirees' circumstances differ, decumulation goals also vary widely among them, depending on their needs, aspirations, and risk aversion. Guyton and Klinger (2006) reported that most retirees have the following goals for income withdrawal from their investment:

- maximize income withdrawals;
- minimize the possibility of running out of funds within the target time frame;

- protect purchasing power from inflation;
- have a steady stream of income and avoid undesired changes such as cutting or freezing withdrawals.

Meeting all four goals simultaneously is a very difficult task and, in addition, preferences for bequests vary considerably from country to country based on different cultures and tax policies. When we consider a decumulation investment strategy, we focus on three main objectives: address liquidity needs for unforeseen expenses, secure basic consumption needs until death, and transfer capital to bequests.

In addition, we face different risks during the accumulation and decumulation phases. As saving patterns in the accumulation phase follow a dollar-cost averaging strategy (Graham, 2006), savers are more resistant to short-term price (market) risk. During the accumulation phase, we usually save a fixed amount of money regularly and invest it in stocks, bonds, mutual funds, and other assets. When financial markets fall, we can invest in these financial instruments at cheaper prices, which means we can buy more shares with the same amount of savings. This strategy allows savers to invest at a lower overall cost over the long term and keeps their money working consistently, which is a key factor in long-term investment growth. However, during the decumulation phase, we have a regular income withdrawal which results in a permanent loss of capital when financial markets fall. It is therefore a very difficult task for portfolio managers to find the right balance between income withdrawals and investment strategies that meet the needs of retirees and protect capital from market volatility. When applying the decumulation investment strategy, retirees are exposed to a variety of risks, including:

- **Longevity risk**
  Longevity risk, as often used by the insurance industry, refers to the risk that life expectancy exceeds pricing assumptions, resulting in increased cash flow requirements for the insurance company or pension fund. By analogy, longevity risk in decumulation investment strategies refers to the risk of retirees living beyond their current expectations and not having enough money to sustain themselves in later life. However, the decumulation investment strategies are not mutual, which means that retirees have their own accounts and are exposed to their own longevity risk rather than sharing longevity risk with others, as is the case with annuities or tontines. This risk can be assessed by retirees based on their health status, and they can change their investment horizon and the amount they spend during the decumulation phase.

- **Sequencing risk**
  Sequential risk is the risk that the timing of withdrawals from retirement accounts is unfavorable, resulting in a reduction in their retirement funds. During the retirement phase, we need to determine the allocation of different assets so that the portfolio is well diversified. However, any type of investment strategy is exposed to market risk. Financial markets can go down at any time, but retirees prefer the stability of income withdrawals. Hence, there is a mismatch between market risk and income withdrawal. In this case, the timing of market risk will significantly affect retirees' portfolios. For example, when retirees' withdrawals are continuous and fixed in amount, a market

decline that occurs early in retirement will have a greater impact than a decline of the same level that occurs later. For some retirees, they will have to draw regular income from their pension portfolios to meet their living needs, regardless of the performance of the underlying investments. In such cases, the sequencing risk is even more acute.

- **Inflation risk**
  As described previously in this section, one of the primary objectives for most retirees is to protect their purchasing power from inflation. Inflation risk is often defined as higher-than-expected increases in the price of goods and services. It is worth noting that retirees spend differently from employees in that they spend more on certain goods and services, such as travel in early retirement, luxury items, and health care at the end of life. In contrast, they do not need to spend as much on housing and education. Therefore, we need to use appropriate methodologies to properly measure the inflation risk of retirees.

2.2 Overview of decumulation strategies

At retirement, retirees need to develop a long-term plan to gradually spend down their assets and have generally three options to deal with their savings:

- withdraw all savings at once;
- purchase a life annuity[2] and receive regular payments until death;
- continue to invest savings in the financial market while allowing to draw from savings at any time.

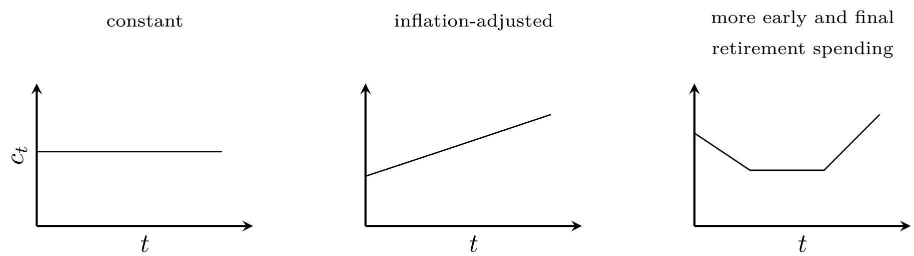
Following the third option, the decumulation investment strategy is a strategic way to convert retirees' pension savings into retirement income to ensure regular payments for the rest of their lives. As described in Bernhardt and Donnelly (2018), the process of decumulation consists of income withdrawal and investment strategies:

- **How much can be withdrawn on a regular basis to meet the basic expenditure needs of retirees?**
  Deciding the amount of income withdrawal is a very challenging task for retirees, requiring comprehensive consideration of long-term risks such as inflation, increased life expectancy, and health risks. The consumption needs of retirees may also vary due to differences in financial situations and pre-retirement spending levels. For simplicity, they can set the consumption demand as a constant or consider the level of inflation. In addition, they can further customize the consumption demand, as shown in the famous U-shaped diagram in Figure 2: in the early stages of retirement, they spend more because they are more active and keen to travel. After that, they enter a phase of low spending, yet in later years, spending increases significantly again due to the cost of a health care.

- **How do we invest the remaining funds?**
  A decumulation investment strategy can be a one-off decision taken at the point of retirement or it could also be a sequence of actions taken over time. For example,

[2] For a more detailed explanation of life annuities and tontines, see Appendix A.2 and A.3.

Figure 2: Spending needs in retirement



Source: Amundi Investment Institute

retirees may choose a constant mix of 60% stocks and 40% bonds or rebalance their portfolio based on market conditions, their remaining wealth, and income withdrawal needs.

Both life annuities and decumulation strategies are designed to draw down capital gradually during retirement. In addition, decumulation strategies have three other advantages: flexibility, especially to recover capital in the event of unforeseen expenses; allowing to bequeath capital; continuing to invest in risky assets in financial markets to achieve portfolio appreciation. However, unlike life annuities, retirees can not share the risk with others when using a decumulation strategy. They will continue to withdraw their pension from the fund until it runs out. They thereby run the risk of using up all of their money before they die. Thus, we need to find an appropriate amount that will allow retirees to withdraw funds from their portfolios without emptying them prematurely, and we need to find the right balance between utilizing risky assets for portfolio appreciation and avoiding the risk of running out of funds within the target time frame. In addition to that, we also need to consider several constraints, such as the retiree's risk profile, e.g., the retiree's risk appetite for the maximum allocation of risky assets, and bequests. As described in Fullmer (2008), the opinions of retirees on bequests vary. Some may cut costs and preserve assets for future generations, while others are dismissive of bequests. In summary, the decumulation strategy is a very complex problem because it requires a long-term perspective that takes into account factors such as the time value of money, inflation, personal health, potential care costs, willingness to bequeath, taxes, unpredictable expenses, and market risk.

2.3 The safe withdrawal rate method

Planning for retirement income withdrawal is a critical aspect of financial management in retirement. However, the task is far from straightforward, and the advice offered to retirees often relies on simplistic rules. In the subsequent section, we quickly introduce the evolution of these approaches within the financial advisor community.

A simplistic approach to managing retirement wealth involves dividing the total sum into equal portions and withdrawing one portion annually. The primary drawback of this strategy lies in the fact that the remaining wealth is not actively invested in the financial markets to capitalize on compound interest and potential risk premiums, which can significantly augment the overall wealth. Conversely, the strategy's advantage lies in its ease of comprehension and the retiree's ability to independently execute it. Building upon this basic strategy, a better approach can be considered. The retiree withdraws a fixed amount annually and invests the remainder in the financial market. As explained in Section 2.2, retirees face two crucial decisions. Firstly, they must determine how to allocate their wealth, including which assets to invest in and the allocation amounts for each asset. Secondly, they need to decide on the annual withdrawal amount, with the key objective of ensuring that his savings are sufficient to support his financial needs throughout retirement.

One of the earliest and most influential contributions to this field came from Bengen (1994, 1996, 1997). His work is an extension of the previous work of Bierwirth (1994), which introduced the concept of evaluating retirement plans based on historical scenarios rather than relying solely on long-term averages. Bengen's research methodology involved employing bootstrap simulations utilizing empirical data from the United States spanning the period from 1927 to 1994. Initially, the author assessed the survivability of payments within a simple 50/50 portfolio of stocks and bonds over a 35-year horizon to determine the maximum safe withdrawal rate (SWR), accounting for inflation. The results of the simulation led Bengen to conclude that an inflation-adjusted withdrawal rate of 4% relative to the initial wealth could be considered a prudent and safe choice. Furthermore, Bengen extended his investigation to various constant mixed portfolios. His findings indicated that an excessive allocation to either too many or too few risky assets could adversely impact the portfolio's long-term viability. Consequently, Bengen recommended asset allocations within the range of 50% to 75% in favor of equity holdings.

This particular rule, now widely recognized within the financial advisor community as the 4% rule, should be noted for its nomenclature. The number 4% within the rule is not an absolute indicator but rather a guideline based on the financial advisor's assessment and future projections of market conditions, encompassing financial and macroeconomic factors such as interest rates, risk premiums, inflation, and more. For example, Table 2 shows the SWR based on Morningstar's 30-year expectation of stock-bond constant mixed portfolio as of December 2022. The figures in this table represent a prudent choice of the safe withdrawal rate under each investment horizon and each constant-mix allocation to ensure that the fund does not run out of money prematurely.

Table 2: 30-year starting safe withdrawal % by asset allocation for 90% success rate

equity weighting	10 years	20 years	30 years	40 years
100	8.5	4.7	3.5	3.1
80	8.9	4.9	3.7	3.1
60	9.3	5.2	3.8	3.2
40	9.6	5.3	3.8	3.2
20	9.7	5.2	3.7	3.0
0	9.4	4.8	3.3	2.6

Source: Morningstar (2022)

Bengen's work serves as a foundational basis for subsequent studies in the field. Cooley *et al.* (1998) adopted a similar methodology in their studies and largely confirmed Bengen's findings. Their research underscores the significance of maintaining a moderate to high level of exposure to risk within portfolios, as well as the prudence of implementing a cautious withdrawal rate typically falling within the range of 3% to 4% for the sustainability of the retirement income strategy. Building on this finding, researchers such as Pfau (2010) conducted a global analysis of 17 developed countries. Their study revealed that nations with a history of lower equity returns experienced safe withdrawal rates below the commonly cited 4% benchmark. Notably, countries like Germany and Japan exhibited safe withdrawal rates as low as 1.14% and 0.47%, respectively, during the period spanning from 1926 to 2008.

Another research path focuses on tailoring retirement strategies to accommodate individual client preferences. For instance, Bengen (1996) explores the concept of time-based risk-averse utility for retirees by introducing a gradual transition to bonds over time as clients age. The findings suggest that, for a 30-year payment horizon, a 4% rule with an initial exposure to risk assets ranging from 50% to 70% could still be deemed safe. In a subsequent study of Bengen (2001), more flexible payment strategies are introduced. These include the "*prosperous retirement*" approach, which entails varying consumption levels in different periods, and the "*performance-adjusted withdrawal*", where the withdrawal amount depends upon the fund's performance. The latter approach aligns with the work of Merton (1969), which will be discussed in more detail in Section 2.4.2. Zolt (2013) modifies the SWR rule by incorporating withdrawals below the inflation-adjusted rate. This strategy alleviates the pressure on payment liabilities, thereby allowing retirees to increase their initial withdrawal rates. Other withdrawal strategies are also introduced in Blanchett *et al.* (2012) or Morningstar (2022). It is important to highlight that the objective of these more flexible payment strategies extends beyond accommodating client preferences. They also aim to enhance the overall sustainability of the retirement strategy and the total income of the retiree.

The second facet of the SWR is asset allocation, with various strategies compared to constant-mix allocation being studied. One such strategy, introduced by Bengen (1996), involves a gradual transition towards bonds over time. Additionally, in Blanchett (2007), a comprehensive examination of different glide paths, including constant-mix, constant piecewise, linear decreasing, convex and concave decreasing, was undertaken. The result reinforces the effectiveness of a constant allocation strategy as the alternative strategies hardly surpass its performance. Consequently, the author recommended a balanced portfolio, such as a 60/40 allocation, as the most prudent choice. In Estrada (2016), encompassing 19 developed countries with data spanning from 1900 to 2009, similarly advocated for a 60/40 allocation as a viable solution. However, in a later study conducted, Pfau and Kitces (2014) stumbled upon a surprising and counter-intuitive result. Their results showed that glide paths involving an increasing exposure to risky assets over time improved the survivability of retirement income strategies. In this study, we will provide a clearer elucidation of this phenomenon.

Overall, the literature on retirement strategy has evolved from Bengen's pioneering work in the early 1990s to a large body of practical research that addresses diverse aspects of retirement income planning, emphasizing the significance of asset allocation, withdrawal rate strategies, and the importance of adapting to changing circumstances throughout retirement. These studies collectively serve as a main guideline for financial advisors to help clients secure and prosperous retirements. Readers can find a more comprehensive review of the SWR rule and a recommendation of the rules for practitioners in Kitces (2014). In summary, the safe withdrawal rate rule, often referred to as the 4% rule, entails utilizing a conventional constant-mix portfolio of stock and bond (with a balance to moderate allocation risky assets like 50/50, 60/40, 70/30) along with an initial withdrawal rate of $x\%$. This withdrawal rate can be adjusted dynamically throughout the retirement period to enhance the strategy's sustainability.

2.4 Optimal control theory and Hamilton-Jacobi-Bellman equation

From a mathematical point of view, the investment-consumption problem for one retiree during the decumulation phase can be considered as an optimal control problem, i.e., to find the best way to control a dynamic system over a certain period $[0, T]$ in order to optimize an objective function. In our case, this dynamic system refers to the retiree's portfolio's wealth X_t, which invests in different types of financial assets in the financial market, and the control vector u_t at each time t includes all decision variables, such as the consumption and the allocation of the portfolio. We also need to introduce a utility function U to represent the retiree's preferences or risk aversion. By convention, we want to maximize our objective function, which is usually in the form of an integral as follows:

$$\int_0^T U\left(X_t, u_t, t\right) \mathrm{d}t$$

where u_t is the control vector at time t and X_t is the system state vector at time t. Then, we define the value function $V(X_t, t)$ as follows:

$$V\left(X_t, t\right) = \max_u \left\{ \int_t^T U\left(X_t, u_t, t\right) \mathrm{d}t + \tilde{U}\left(X_T, T\right) \right\} \tag{1}$$

where $\tilde{U}\left(X_T, T\right)$ gives the bequest value of the utility function at the final state. Then, the optimal control problem consists of finding optimal controls u_t for each time t and this problem can be solved by applying Bellman's principle of optimality in the theory of dynamic programming. The basic idea of this approach is to consider a family of optimal control problems with different initial times and states and to determine the relationship between these problems using a non-linear partial differential equation, the so-called Hamilton-Jacobi-Bellman (HJB) equation. If the HJB equation can be solved analytically or numerically, then we can obtain solutions to the entire family of problems with different initial times and states, and thus the solution of the original problem. In addition, we can also generalize deterministic optimal control problems to stochastic optimal control problems:

$$V\left(X_t, t\right) = \max_u \mathbb{E}\left[\int_t^T U\left(X_t, u_t, t\right) \mathrm{d}t + \tilde{U}\left(X_T, T\right) \right] \tag{2}$$

In particular, the HJB equation involves a first-order PDE in the deterministic case and a second-order PDE in the stochastic case. According to Bernhardt and Donnelly (2018), the terminal time T could be a random time of death or a fixed time, depending on the problem considered. In our case, since we are considering a separate decumulation strategy for each individual, we do not need to consider the life expectancy of the retirees. Each retiree has his or her own retirement savings account and can set different investment horizons T according to their circumstances. Furthermore, the above problem assumes that the retiree derives some utility from the remaining wealth at the terminal date T. If he or she doesn't want to leave some bequest at the end of the investment horizon, we can also remove the term $\tilde{U}(X_T, T)$ from the objective function, then the optimal consumption and investment plan will end up with exactly zero final wealth.

2.4.1 Financial market models

As we explained in the previous section, an optimal control problem is a dynamic system that needs to be modeled. To simplify the simulation, we usually consider that the retiree's portfolio is invested in two financial assets: a risk-less asset with a constant instantaneous rate of return $r \geq 0$, and a risky asset whose price follows a geometric Brownian motion with constant volatility $\sigma > 0$ and drift μ. Therefore, the price B_t of the risk-free financial asset at time t with the deterministic interest rate r_t is described as follows:

$$\frac{\mathrm{d}B_t}{B_t} = r_t \,\mathrm{d}t \qquad B_t = B_0 e^{\int_0^t r_s \,\mathrm{d}s}$$

When the interest rate is constant, we have $r_t = r$, $\forall t$, then

$$B_t = B_0 e^{rt} \tag{3}$$

And we suppose that the price S_t of the risky asset at time t satisfies the following stochastic differential equation (SDE):

$$\frac{\mathrm{d}S_t}{S_t} = \mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t$$

When the drift μ_t and the volatility σ_t are constant, we have $\mu_t = \mu$ and $\sigma_t = \sigma$, $\forall t$, then

$$S_t = S_0 e^{\left(\mu - \sigma^2/2\right)t + \sigma W_t} \tag{4}$$

We suppose that retirees may invest a proportion α_t of their wealth in the risky asset at time t and the rest $1 - \alpha_t$ in the risk-free asset. Therefore, the dynamics of the fund wealth X_t is described by the following equation:

$$\begin{aligned} \mathrm{d}X_t &= X_t \left(\alpha_t \frac{\mathrm{d}S_t}{S_t} + (1 - \alpha_t) \frac{\mathrm{d}B_t}{B_t} \right) - c_t \,\mathrm{d}t \\ &= \left[(r + \alpha_t (\mu - r)) X_t - c_t \right] \mathrm{d}t + \alpha_t \sigma X_t \,\mathrm{d}W_t \end{aligned} \tag{5}$$

where $c_t \,\mathrm{d}t$ is the amount withdrawn by retirees between t and $t + \mathrm{d}t$.

In addition, Equation 5 can be extended to a multi-variate case, i.e., retirees may invest in n correlated risky assets and each follows the following dynamic process:

$$\mathrm{d}S_t^i = \mu_i S_t^i \,\mathrm{d}t + \sigma_i S_t^i \,\mathrm{d}W_t^i$$

where the Brownian motions $\{W_t^i\}_{i=1}^n$ are correlated such that $\mathrm{E}\left(\mathrm{d}W_t^i \mathrm{d}W_t^j\right) = \rho_{i,j}\,\mathrm{d}t$ and $\rho_{i,i} = 1$. Then, the wealth of the fund evolves as:

$$\begin{aligned} \mathrm{d}X_t &= X_t \left(\alpha_t^\top \left(\operatorname{diag}\left(S_t\right)\right)^{-1} \mathrm{d}S_t + \left(1 - \alpha_t^\top \mathbf{1}_n\right) \frac{\mathrm{d}B_t}{B_t} \right) - c_t\,\mathrm{d}t \\ &= \left[\left(r + \alpha_t^\top \left(\mu - r\mathbf{1}_n\right)\right) X_t - c_t \right] \mathrm{d}t + \alpha_t^\top \sigma X_t\,\mathrm{d}W_t \end{aligned} \tag{6}$$

where $\mathrm{d}W_t = \left(\mathrm{d}W_t^1, \cdots, \mathrm{d}W_t^n\right)$ and $\mathbf{1}_n = (1, \cdots, 1)$.

2.4.2 Maximizing the expected discounted utility of lifetime consumption

One way of determining a retiree's consumption and investment strategy during retirement is to maximize the expected discounted utility of future lifetime consumption, which is also known as Merton's optimal investment-consumption problem in continuous-time finance. Merton (1969) proposed a classical framework to deal with the investment-consumption problem during the decumulation phase, where retirees need to decide how much to consume and how to allocate their remaining wealth between stocks and a risk-free asset in order to maximize the expected discounted utility of all future consumption. One of the main features of his formulation is that time is continuous, which implies that the evolution of the portfolio's wealth follows a continuous stochastic process and that controls such as consumption and asset allocation are also time-continuous.

Let us consider a retiree over a fixed time interval $[0, T]$, where the time 0 refers to his or her date of retirement and the time T is a given investment horizon, e.g., 30 years. We denote by X_0 the savings of the retiree during the accumulation phase and by X_t the portfolio's wealth at time t. We assume that this retiree withdraws consumption c_t at time t and has two assets to invest in the financial market for the remaining wealth: a risk-free asset B_t and a risky asset S_t, the dynamics of which are described by Equations 3 and 4, respectively. Then, the decumulation problem for this retiree is to decide the amount of consumption c_t and the proportion of wealth α_t for the risky asset at each time t. The portfolio is fully invested, which means that a proportion α_t of his wealth is invested in the risky asset, and the rest $1-\alpha_t$ is invested in the risk-free asset. As we have explained in the previous section, the dynamics of the portfolio's wealth X_t is described by the following equation:

$$\mathrm{d}X_t = \left[\left(r + (\mu - r)\,\alpha_t\right) X_t - c_t\right] \mathrm{d}t + \sigma \alpha_t X_t\,\mathrm{d}W_t \tag{7}$$

where r, μ and σ are constants. It is worth noting an important constraint that consumption cannot be negative, but α_t is unrestricted, i.e. borrowing or short selling is allowed. Thus, we have $c_t \geq 0, \forall t \geq 0$.

Based on economic theory, we need to introduce the concept of "*utility*" to evaluate the retiree's investment-consumption strategies. The goal of the retiree is to choose an optimal strategy that maximizes the expected total discounted utility of consumption over $[0, T]$, which is described by the following formula:

$$\mathbb{E}\left[\int_0^T e^{-\rho t} F\left(t, c_t\right) \mathrm{d}t + e^{-\rho T} G\left(X_T\right) | X_0 = x_0\right]$$

where $F(t, c_t)$ is the utility function of consumption and $G(X_T)$ is the utility function of the remaining money at the terminal date T. These utility functions F and G represent the retiree's consumption preference or risk aversion and should satisfy some standard properties, such as increasing and concave.

Theoretically, an individual's utility for current consumption is higher than their utility for future consumption, so we need to introduce a discount rate $e^{-\rho t}$ for the utility functions $F(t, c_t)$ and $G(X_T)$. As explained in Rao and Jelvis (2022), we can consider the fund wealth X_t as a dynamic system, and the continuous-time Markov decision process for this system is defined by Equation 5. The state of this dynamic system is described by (t, X_t), and at each time t, we can take action c_t and α_t to decide the consumption and how to invest the rest money. These actions update the state of the system, then we decide on new actions based on the new state, and so on. Our goal is to find the Policy: $(t, X_t) \to (c_t, \alpha_t)$ that maximize the expected discount utility of lifetime consumption, which includes the rest of the money at the terminal date T. In our case, the retiree's decumulation problem can be described as a classical stochastic optimal control problem as follows:

$$\begin{gathered}\max_{c_t,\alpha_t} \mathbb{E}\left[\int_0^T e^{-\rho t} F(t, c_t)\,\mathrm{d}t + e^{-\rho T} G(X_T) \,|X_0 = x\right] \\ \mathrm{d}X_t = \left[\left(r + (\mu - r)\,\alpha_t\right) X_t - c_t\right]\mathrm{d}t + \sigma\alpha_t X_t\,\mathrm{d}W_t \\ c_t \geq 0,\ \forall t \geq 0\end{gathered} \tag{8}$$

As detailed by Björk (2020), the idea of solving Problem 8 is to embed our original stochastic control problems into a larger class of problems, all of which can then be described by the Hamilton-Jacobi-Bellman equation. To simplify the notation, we rewrite the dynamic of X_t as follows:

$$\begin{aligned}\mathrm{d}X_t &= \tilde{\mu}(t, X_t, c_t, \alpha_t)\,\mathrm{d}t + \tilde{\sigma}(t, X_t, c_t, \alpha_t)\,\mathrm{d}W_t \\ &= \tilde{\mu}\mathrm{d}t + \tilde{\sigma}\mathrm{d}W_t\end{aligned}$$

and we define the value function $V(t, X_t)$ at time t:

$$\begin{aligned}V(t, X_t) &= \mathbb{E}\left[\int_t^T e^{-\rho(s-t)} F(s, c_s)\,\mathrm{d}s + e^{-\rho(T-t)} G(X_T) \,|X_t\right] \\ &= \mathbb{E}_t\left[\int_t^T e^{-\rho(s-t)} F(s, c_s)\,\mathrm{d}s + e^{-\rho(T-t)} G(X_T)\right]\end{aligned}$$

Thus, the optimal value function $V^*(t, X_t)$ is:

$$V^*(t, X_t) = \max_{c_t,\alpha_t} V(t, X_t)$$

Using the principle of dynamic programming, we can obtain the HJB equation[3] associated

[3]The detailed mathematical derivation can be found on page 52 and is included in the Appendix A.4 for the sake of completeness.

with our stochastic optimal control problem (Problem 8):

$$\frac{\partial V^*}{\partial t} + \max_{c_t, \alpha_t} \left[F(t, c_t) + \tilde{\mu} \frac{\partial V^*}{\partial x} + \frac{1}{2} \tilde{\sigma}^2 \frac{\partial^2 V^*}{\partial x^2} \right] = \rho V^* \tag{9}$$

Equation 9 is a non-linear second-order partial differential equation and solving the retiree's utility maximization problem (Problem 8) is equivalent to solving this partial differential equation. In particular, we have:

$$\begin{cases} \tilde{\mu} = \tilde{\mu}(t, X_t, \alpha_t, c_t) = (r + (\mu - r)\alpha_t) X_t - c_t \\ \tilde{\sigma} = \tilde{\sigma}(t, X_t, \alpha_t) = \sigma \alpha_t X_t \end{cases} \tag{10}$$

and the terminal condition is:

$$V^*(T, X_T) = G(X_T) \tag{11}$$

One of the common choices of the utility functions for $F(t, c_t)$ is the constant relative risk aversion (CRRA) form:

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \text{for } 0 < \gamma \neq 1 \text{ and } c \geq 0$$

where the constant γ refers to the investor's risk aversion, i.e., the tendency to prefer outcomes with low uncertainty to those with high uncertainty, even if the average outcome of the latter is equal to or higher than that of the outcome with high certainty. Thus, the higher the gamma, the more risk-averse the retiree will be, the more cautious he or she will be in taking risks and may be reluctant to own risky assets. In addition, we have $\gamma = -c \cdot \frac{U''(c)}{U'(c)}$, where $-\frac{U''(c)}{U'(c)}$ is the Arrow-Pratt measure of relative risk aversion. For the special case $\gamma = 1$, $U(c) = \log(c)$. For the utility function $G(X_T)$ at the terminal date T, we can also consider a CRRA form with a parameter ϵ^γ, which reflects the desired level of bequest X_T. In practice, we can set ϵ to a small positive value, i.e. $0 < \epsilon \ll 1$ to reflect the "*no desired bequest*" case. According to Rao and Jelvis (2022), this $\epsilon-$formulation is necessary for technical reasons. Thus, we have:

$$\begin{aligned} F(t, c_t) &= \frac{c_t^{1-\gamma}}{1-\gamma} \\ G(X_T) &= \epsilon^\gamma \frac{X_T^{1-\gamma}}{1-\gamma} \end{aligned} \tag{12}$$

Analytical solution To find the analytical solution to Equation 9 in the case where utility functions are defined as Equation 12, we follow the mathematical derivation of Rao and Jelvis (2022). For the sake of simplicity, we let α_t be unconstrained and we will see that this simplification will not change the final solution. We note:

$$\Phi(t, X_t, \alpha_t, c_t) = F(t, c_t) + \tilde{\mu}(t, X_t, \alpha_t, c_t) \frac{\partial V^*}{\partial x} + \frac{1}{2} \tilde{\sigma}(t, X_t, \alpha_t)^2 \frac{\partial^2 V^*}{\partial x^2}$$

We have

$$\frac{\partial V^*}{\partial t} + \max_{c_t, \alpha_t} \Phi\left(t, X_t, \alpha_t, c_t\right) = \rho V^* \tag{13}$$

We can then find the optimal values of $c_t^\star, \alpha_t^\star$ by taking the partial derivatives of Φ with respect to c_t and α_t, and equating them to 0 to satisfy the first-order optimality condition:

- Partial derivative of Φ with respect to c_t:

$$\frac{\partial \Phi}{\partial c_t} = \frac{\partial F}{\partial c_t} + \frac{\partial \tilde{\mu}}{\partial c_t}\frac{\partial V^*}{\partial x}$$

- Partial derivative of Φ with respect to α_t:

$$\frac{\partial \Phi}{\partial \alpha_t} = \frac{\partial \tilde{\mu}}{\partial \alpha_t}\frac{\partial V^*}{\partial x} + \tilde{\sigma}\frac{\partial \tilde{\sigma}}{\partial \alpha_t}\frac{\partial^2 V^*}{\partial x^2}$$

In our case,

$$F\left(t, c_t\right) = \frac{c_t^{1-\gamma}}{1-\gamma}$$
$$\tilde{\mu}\left(t, X_t, \alpha_t, c_t\right) = \left(r + (\mu - r)\alpha_t\right) X_t - c_t$$
$$\tilde{\sigma}\left(t, X_t, \alpha_t\right) = \sigma \alpha_t X_t$$

Then, we have:

$$c_t^{*\,-\gamma} - \frac{\partial V^*}{\partial x} = 0$$
$$\Rightarrow \quad c_t^* = \left(\frac{\partial V^*}{\partial x}\right)^{-\frac{1}{\gamma}}$$

and

$$(\mu - r)\frac{\partial V^*}{\partial x} + \sigma^2 \alpha_t^* X_t \frac{\partial^2 V^*}{\partial x^2} = 0$$
$$\Rightarrow \quad \alpha_t^* = \frac{-(\mu - r)\frac{\partial V^*}{\partial x}}{\sigma^2 X_t \frac{\partial^2 V^*}{\partial x^2}}$$

We replace the term c_t^* and α_t^* in Equation 9 and derive the expression:

$$\frac{\partial V^*}{\partial t} - \frac{(\mu - r)^2}{2\sigma^2}\frac{\left(\frac{\partial V^*}{\partial x}\right)^2}{\frac{\partial^2 V^*}{\partial x^2}} + rX_t\frac{\partial V^*}{\partial x} + \frac{\gamma}{1-\gamma}\left(\frac{\partial V^*}{\partial x}\right)^{\frac{\gamma-1}{\gamma}} = \rho V^* \tag{14}$$

and the terminal condition for this partial differential equation is:

$$V^*\left(T, X_T\right) = \epsilon^\gamma \frac{X_T^{1-\gamma}}{1-\gamma}$$

We can surmise with a guess solution in terms of a deterministic function f of time:

$$V^*\left(t, X_t\right) = f\left(t\right)^\gamma \frac{X_t^{1-\gamma}}{1-\gamma}$$

Substituting the guess solution into Equation 14, we obtain a simple ordinary differential equation:

$$f'(t) = \frac{\rho - (1-\gamma) \cdot \left(\frac{(\mu-r)^2}{2\sigma^2\gamma} + r \right)}{\gamma} f(t) - 1 \tag{15}$$

Using the ϵ-formulation provides a simple boundary condition for Equation 15:

$$f(T) = \epsilon$$

Then, we deduce that:

$$f(t) = \begin{cases} \frac{1+(\nu\epsilon-1)e^{-\nu(T-t)}}{\nu} & \text{for } \nu \neq 0 \\ T - t + \epsilon & \text{for } \nu = 0 \end{cases}$$

where

$$\nu = \frac{\rho - (1-\gamma) \cdot \left(\frac{(\mu-r)^2}{2\sigma^2\gamma} + r \right)}{\gamma}$$

Finally[4],

$$\alpha_t^* = \frac{\mu - r}{\sigma^2 \gamma} \tag{16}$$

$$c_t^* = \begin{cases} \frac{\nu X_t}{1+(\nu\epsilon-1)e^{-\nu(T-t)}} & \text{for } \nu \neq 0 \\ \frac{X_t}{T-t+\epsilon} & \text{for } \nu = 0 \end{cases} \tag{17}$$

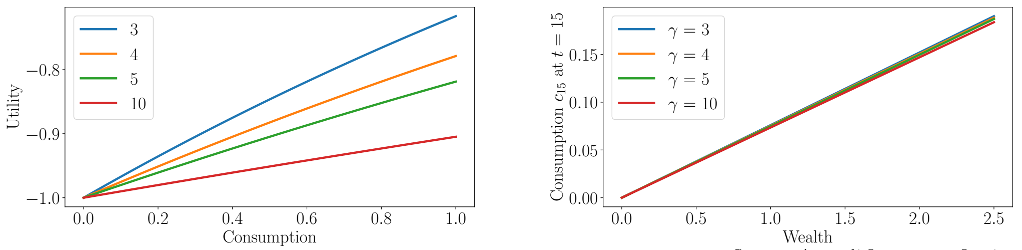
Equation 16 shows that the optimal investment α_t^* is a constant-mix portfolio allocation in the Merton framework, and Equation 17 shows that the optimal consumption c_t^* is proportional to X_t. In other words, it raises consumption when the portfolio performs well and lowers it when the portfolio underperforms. The advantage of this type of strategy is that the funds never run out, but the consumption is unstable.

Impact of risk aversion parameters In this section, we want to assess the impact of the risk aversion parameter on the optimal policy. To this end, we simulate 1 000 scenarios for the risky asset with performance μ and volatility σ of 5% and 15% respectively, and we set both the risk-free rate r and the discount factor ρ to 1%. For simplicity, we assume that the initial wealth of the portfolio is 1 and we set the investment horizon at 30 years. According to Equation 16, the use of different risk aversion parameters γ implies the choice of different constant portfolio investment strategies. If we take $\gamma = 3, 4, 5, 10$, then the corresponding risky asset allocation is $\alpha_t = 59.3\%, 44.4\%, 35.6\%, 17.8\%$. As shown in Figure 3a, when γ is large, the investor prefers a low-return and low-risk decision. Nevertheless, as can be seen in Figures 3a and 3b, the effect of the risk aversion parameter is not significant. For different values of γ, while there is a significant effect on the optimal constant-mix allocation, there is little effect on consumption, which is more influenced by the value of portfolio wealth. The distribution and first moments of total wealth are shown in Figure 4 and Table 3.

[4] The detailed mathematical derivation can be found on page 53 and is included in the Appendix A.5 for the sake of completeness.

Figure 3: Impact of risk aversion parameter γ

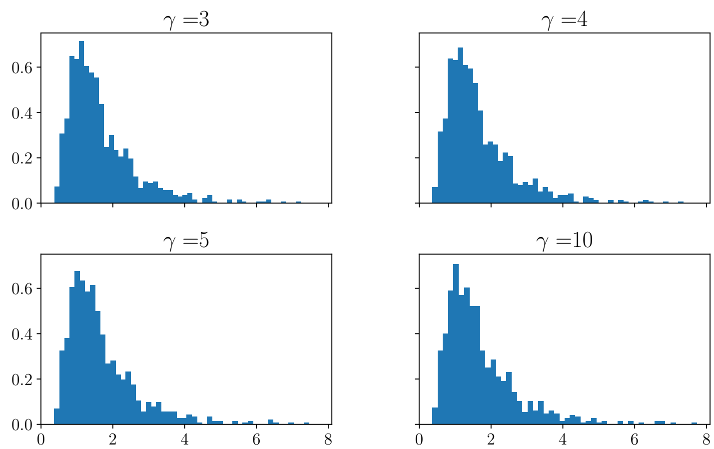
(a) CRRA utility function (b) Optimal consumption



Source: Amundi Investment Institute

Figure 4: Impact of risk aversion parameter γ (Finite investment horizon)

Total wealth (all consumption and bequest)



Source: Amundi Investment Institute

As shown in Figure 4, the impact of the risk aversion parameter is smaller in the case of a finite investment horizon.

Table 3: Characteristics of the total wealth distribution based on 1 000 simulations

	$\gamma = 3$	$\gamma = 4$	$\gamma = 5$	$\gamma = 10$
Mean:	1.66	1.67	1.68	1.70
Median:	1.40	1.41	1.41	1.43
Standard deviation:	0.95	0.96	0.98	1.01
Skewness:	1.88	1.90	1.91	1.96
Kurtosis:	5.01	5.13	5.22	5.48

Source: Amundi Investment Institute

From Equations 16 and 17, we get the solution to Merton's portfolio problem and we notice that α_t^* is a constant that is independent of the time t, the investment horizon T and the value of the portfolio's wealth X_t. This reflects the fact that retirees should always follow a constant-mix strategy to invest between risk-free and risky assets, no matter what the wealth of the portfolio is or how close we are to the investment horizon. This conclusion remains the same for the case of n risky assets. However, the consumption c_t^* is a function of the value of the remaining wealth X_t and how far we are from the terminal date $T - t$, meaning that the withdrawal of income is hardly stable and cannot be planned. Optimal consumption may vary considerably from one period to another, as it depends on the level of remaining wealth in the portfolio, which in turn is strongly influenced by the economic situation in that period. As explained in Section 2.2, one of the key objectives of retirees is to have a steady stream of income and to avoid undesired changes such as cutting or freezing withdrawals. Their affordability to adjust consumption in difficult times is limited, especially for some retirees who need a certain amount of payment to meet their most basic needs. Therefore, designing a decumulation strategy by maximizing the expected discounted utility of lifetime consumption can provide a simple analytical solution to the dilemma between consumption and investment, but this approach is not suitable for practical situations. One way to address this problem is a subsistence level $\tilde{c}$ for the consumption, i.e. $c_t \geq \tilde{c}$ in order to meet the basic needs of retirees. Gong and Li (2006) offers some important insights in this regard. They have studied the role of index bonds in the optimal consumption and portfolio selection problem with CRRA utility function and a subsistence consumption constraint. Another solution is to introduce habit formation in the objective function to limit fluctuations in consumption over time:

$$U(c_t, h_t) = \frac{(c_t - h_t)^{1-\gamma}}{1-\gamma}$$

where the habit level h_t, which represents the standard of living of a retiree, is defined as:

$$h_t = e^{-at} h_0 + b \int_0^t e^{a(s-t)} c_s \, \mathrm{d}s \tag{18}$$

where h_0, a and b are non-negative constants. h_0 is the initial habit level, a and b are persistence and scaling parameters, respectively. As shown in Equation 18, h_t is an exponentially weighted average of past consumption, which implies a higher weighting of recent consumption. Then, the value function to maximize can be written as:

$$\mathbb{E}\left[\int_0^T e^{-\rho t} U(c_t, h_t) \, \mathrm{d}t + e^{-\rho T} G(X_T) \,|X_0 = x_0\right] \tag{19}$$

Next, we solve the associated HJB equation to solve Equation 19. Both of these two methods are more difficult to find analytical solutions than in the case where there are no consumption restrictions, and we need to introduce more parameters into the model. In practice, using more parameters in the model means that there is more estimation risk and that the model is less robust to assumption errors.

From the framework of Merton's portfolio problem, we can recognize that a retirement strategy means that at each point in time, we need to find a balance between the consumption we can enjoy now and the risks we will take in the financial markets in the future. When the portfolio performs well, the retiree may have two choices: to get more consumption immediately or to move to safer investments, which implies lower expected returns but more protection. The analytical solution to Merton's problem follows the first choice and shows that the optimal investment strategy under this framework is the constant-mix strategy, which implies that we always take the same proportion of weight between risky and risk-free assets. In other words, we will take the same level of relative risk whether the portfolio's wealth appreciates or depreciates, and adjust our spending based on market conditions. In addition to this, we can design our investment strategy according to the second choice, i.e., fixing our consumption so that when the portfolio performs well, we move to safer investments, and when the portfolio performs poorly, we allocate more risky assets in our portfolios in the expectation of recovering losses. This mechanism has a profit-taking behavior. One way to achieve this is to design our decumulation strategy by considering the expected discounted utility at the ruin date, rather than maximizing the expected discounted utility of lifetime consumption. In this type of strategy, our consumption is determined and we can use an investment strategy that varies with market conditions. We will discuss this strategy in more detail in the next section.

2.4.3 Maximizing the expected utility of the ruin date

As explained in Section 2.2, retirees prefer to have a regular flow of income and avoid undesirable changes. If we use Merton's portfolio optimization framework, the consumption c_t is a dynamic variable, which depends on portfolio wealth X_t and time t. This is not what retirees expect. We can therefore assume that consumption $\bar{c}_t$ is deterministic or planned at time 0 for any time t, and we need to consider another type of utility function that does not involve consumption but some other dynamic variables in order to calculate expected utility, such as the probability of ruin or the ruin date. Then, we follow the same routine of the stochastic control framework in Section 2.4.2 and solve the associated HJB equation.

In this section, we need to first introduce the concept of the ruin date, which represents the stopping time when the wealth value X_t falls below a given threshold $h \geq 0$:

$$\tau_h(t,x) = \inf\{s \geq 0 : X_{t+s} = h | X_t = x\}$$

where the value of h can be set to 0, or a higher value based on the retiree's preference. Our objective is to delay the occurrence of the ruin date τ_h for as long as possible, by controlling the proportion of allocation to risky assets α_t at time t. To achieve this, we have two approaches:

- We can directly minimize the probability of ruin. The problem consists of minimizing the probability of ruin before a horizon, which can be a fixed value T, the infinity $+\infty$, or the random time of death of an individual τ_d in order to incorporate mortality.
  - The infinity
  $$\mathbb{P}[\tau_h < \infty \mid X_t = h]$$
  - A fixed horizon
  $$\mathbb{P}[\tau_h < T \mid X_t = h, T > t]$$
  - The random time of death
  $$\mathbb{P}[\tau_h < \tau_d \mid X_t = h, \tau_d > t]$$

- We can introduce an utility function U for the ruin date τ_h and then maximize the expected utility:

$$V^*(t, X_t) = \sup_{\alpha_t} \mathbb{E}\left[U(\tau_h(t, x)) \mid X_t\right]$$

  The problem consists of maximizing the expected utility of the ruin date. Unlike the utility function of consumption in Section 2.4.2, we don't need an integral that sums total utility up to terminal date T.

The first approach is straightforward, but it is difficult to obtain closed-form solutions. In contrast, we can choose the appropriate utility function in the second approach to help us obtain an easier solution. Therefore, we use the second method in this paper, where our goal is to delay the ruin time τ_h as long as possible. To do this, we need to introduce in our stochastic optimal control problem a utility function $U(\tau_h)$ that depends only on the destruction date, not on consumption. In particular, $U(\tau_h)$ is a monotonically increasing function of τ_h, which means that the later the fund ruin, the better, and $U(+\infty)$ is finite. Therefore, the problem of minimizing the probability of ruin can be described as follows:

$$\begin{gathered}\max_{\alpha_t} \mathbb{E}\left[U(\tau_h(t, X_t)) | X_t\right] \\ \mathrm{d}X_t = \left[(r + (\mu - r)\alpha_t) X_t - \bar{c}_t\right] \mathrm{d}t + \sigma \alpha_t X_t \,\mathrm{d}W_t \\ \bar{c}_t \geq 0, \forall t \geq 0\end{gathered} \tag{20}$$

where $\bar{c}_t$ is the consumption at any time t predetermined at time 0. We define the value function $V(t, X_t)$ at time t:

$$\begin{aligned}V(t, X_t) &= \mathbb{E}\left[U(\tau_h) | X_t\right] \\ &= \mathbb{E}_t\left[U(\tau_h)\right]\end{aligned}$$

Thus, the optimal value function $V^*(t, X_t)$ is:

$$V^*(t, X_t) = \max_{\alpha_t} V(t, X_t)$$

Following the same steps in Section 2.4.2, we can obtain the following HJB equation[5]:

$$\frac{\partial V^*}{\partial t} + \max_{\alpha_t} \left[\tilde{\mu} \frac{\partial V^*}{\partial x} + \frac{1}{2}\tilde{\sigma}^2 \frac{\partial^2 V^*}{\partial x^2}\right] = 0$$

Let

$$\Phi(t, X_t, \alpha_t) = \tilde{\mu} \frac{\partial V^*}{\partial x} + \frac{1}{2}\tilde{\sigma}^2 \frac{\partial^2 V^*}{\partial x^2}$$

Partial derivative of $\Phi(t, x, \alpha_t)$ with respect to α_t:

$$\begin{aligned}\frac{\partial \Phi(t, x, \alpha_t)}{\partial \alpha_t} &= 0 \\ \Rightarrow \alpha_t^* &= \frac{-(\mu - r)\frac{\partial V^*}{\partial x}}{\sigma^2 X_t \frac{\partial^2 V^*}{\partial x^2}}\end{aligned}$$

[5]The detailed mathematical derivation can be found in the Appendix A.7

Finally, we have the HJB equation for this problem:

$$\frac{\partial V^*}{\partial t} - \frac{(\mu - r)^2}{2\sigma^2} \frac{\left(\frac{\partial V^*}{\partial x}\right)^2}{\frac{\partial^2 V^*}{\partial x^2}} + (rX_t - \bar{c}_t) \frac{\partial V^*}{\partial x} = 0 \tag{21}$$

One of the common choices of the utility function for the ruin date is the exponential utility function:

$$U(\tau) = -\exp\left(-\frac{\tau}{\lambda}\right)$$

where λ represents the risk appetite of the investor (i.e. inverse of the risk aversion). The exponential utility is a type of constant absolute risk aversion (CARA) utility function, i.e., its absolute risk aversion coefficient $-\frac{U''(\tau)}{U'(\tau)}$ is a constant:

$$-\frac{U''(\tau)}{U'(\tau)} = -\frac{1}{\lambda}$$

When the value of λ is small, people tend to prefer outcomes with low uncertainty to those with high uncertainty. In addition, this particular family of utility functions possesses a desirable property, that is crucial for ensuring the time stationary of the problem with an infinite horizon, meaning that the control does not depend on time t. In this case, the stochastic optimal control problem is invariant by translation on the temporal axis. In other words, if we start the problem θ years later with the same wealth of fund, then the ruin time will happen θ years later:

$$\begin{aligned}
\tau_h(t+\theta, x) &= \inf\{s \geq t+\theta : X_s = h | X_{t+\theta} = x\} \\
&= \inf\{s \geq t : X_s = h | X_t = x\} + \theta \\
&= \tau_h(t, x) + \theta
\end{aligned}$$

$$\begin{aligned}
V^*(t+\theta, x) &= \max_{\alpha_t} \mathbb{E}\left[U(\tau_h(t+\theta, x)) | X_{t+\theta} = x\right] \\
&= \max_{\alpha_t} \mathbb{E}\left[U(\tau_h(t, x) + \theta) | X_t = x\right] \\
&= \max_{\alpha_t} \mathbb{E}\left[-\exp\left(-\frac{\tau_h(t, x) + \theta}{\lambda}\right) | X_t = x\right] \\
&= \exp\left(-\frac{\theta}{\lambda}\right) \max_{\alpha_t} \mathbb{E}\left[-\exp\left(-\frac{\tau_h(t, x)}{\lambda}\right) | X_t = x\right] \\
&= \exp\left(-\frac{\theta}{\lambda}\right) V^*(t, x)
\end{aligned}$$

We differentiate the relation above with respect to θ to obtain:

$$\begin{aligned}
\frac{\partial V^*}{\partial t}(t+\theta, x) &= \lim_{\theta \to 0} \frac{V^*(t+\theta, x) - V^*(t, x)}{\theta} = \left(\lim_{\theta \to 0} \frac{\exp\left(-\frac{\theta}{\lambda}\right) - 1}{\theta}\right) V^*(t, x) \\
&= -\frac{1}{\lambda} V^*(t, x)
\end{aligned}$$

Therefore we can replace the term $\frac{\partial V^*}{\partial t}$ in the HJB Equation 21. In this case, V^* is independent of time t and is a function that depends only on the portfolio's wealth X_t. Then, we obtain the ordinary differential equation:

$$-\frac{1}{\lambda}V^* + \max_{\alpha_t}\left[\tilde{\mu}\frac{\partial V^*}{\partial x} + \frac{1}{2}\tilde{\sigma}^2\frac{\partial^2 V^*}{\partial x^2}\right] = 0 \tag{22}$$

Wealth objective In our problem, since consumption is constant or predetermined, there exists a value of wealth that ensures infinite payments at each date t, which means $\tau_h = +\infty$. In other words, regardless of whether the investment horizon is infinite or finite, if the wealth of our portfolio exceeds a certain threshold H, after that, we can ensure all future payments simply by investing the portfolio entirely in risk-free assets or buying an annuity. For example, let us consider the case of an infinite investment horizon where we want an annual coupon $c = 5\,000$€ for consumption, and the risk-free rate is assumed to be $r = 1\%$. In this case, when the wealth of the portfolio reaches $H = \frac{c}{r} = 500\,000$€, we only need to invest in risk-free assets to ensure future payments. As the wealth of the portfolio moves away from the threshold, the optimal control invests heavily in risky assets in the expectation of approaching the threshold more quickly. Similarly, the closer the wealth of the portfolio is to the critical value, the smaller the allocation to risky assets. Introducing a finite horizon may introduce a more attainable take-profit behavior. In this case, the threshold varies over time. Given an horizon T, the maximum wealth objective at t is:

$$\begin{aligned} H &= \int_t^T c\exp(-r(s-t))ds \\ &= c\int_0^{T-t}\exp(-rs)ds \\ &= -\frac{c}{r}[\exp(-rs)]_0^{T-t} \\ &= \frac{c}{r}(1-\exp(-r(T-t))) \end{aligned}$$

It is not difficult to conclude that for each date t, the threshold in the case of an infinite investment horizon is much higher than that in the case of a finite investment horizon, as shown in Figure 5. For example, with an interest rate of 1% and a coupon objective of 5 000€, the wealth needed at time 0 to ensure a 30-year payment is $\frac{c}{r}\left(1-\exp^{-rT}\right) \approx 129\,591$€, which is more attainable.

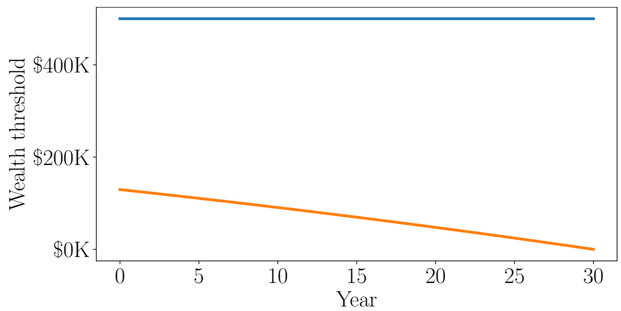
Numerical solution In the general case, the HJB equation that solves the problem of maximizing the expected utility of the ruin date is:

$$\frac{\partial V^*}{\partial t} + \max_{\alpha_t}\left[\tilde{\mu}\frac{\partial V^*}{\partial x} + \frac{1}{2}\tilde{\sigma}^2\frac{\partial^2 V^*}{\partial x^2}\right] = 0 \tag{23}$$

where $\tilde{\mu} = [(r + (\mu - r)\,\alpha_t)\,X_t - \bar{c}_t]$ and $\tilde{\sigma} = \sigma\alpha_t X_t$. To solve Equation 23, we need to introduce the boundary condition in case of the ruin date, i.e. when X_t reaches the threshold h:

$$V^*(t,h) = U(t)$$

Figure 5: Comparison of thresholds in the case of infinite and finite investment horizon (30 Y)



Source: Amundi Investment Institute

and the boundary condition in case of the portfolio's wealth reaches the upper bound H to ensure all future payments without investing in risky assets:

$$V^*(t, H) = U(T)$$

We also need a terminal condition for the time horizon T, where T can be infinite or finite:

$$V^*(T, x) = U(T)$$

Then, we can use an iterative algorithm to solve Equation 23. First, we choose an arbitrary value for all α_t. Then, we solve the differential equation using the finite difference method to get the value of $V^*(t, x)$ for any t and x. We numerically compute the values of $\frac{\partial V^*}{\partial x}$ and $\frac{\partial^2 V^*}{\partial x^2}$ and then find the values of α_t that maximizes $\left[\tilde{\mu}\frac{\partial V^*}{\partial x} + \frac{1}{2}\tilde{\sigma}^2\frac{\partial^2 V^*}{\partial x^2}\right]$. We repeat these steps until convergence.

In the case of the infinity investment horizon, $V^*(t, x)$ has time homogeneity, which means that the translation of the optimal control problem on the time axis is invariant. Then, we have the following relation:

$$-\frac{1}{\lambda}V^* + \max_{\alpha_t}\left[\tilde{\mu}\frac{\partial V^*}{\partial x} + \frac{1}{2}\tilde{\sigma}^2\frac{\partial^2 V^*}{\partial x^2}\right] = 0 \qquad (24)$$

At each iterative step, given α_t, we just need to solve an ordinary differential equation with respect to x. However, if we choose a fixed investment horizon T, the utility function becomes $U(\min(\tau_h, T_{\max}))$, then $V^*(t, x)$ does not have time homogeneity. In this case, at each iterative step, we need to solve a partial differential equation with respect to t and x.

In practice, since we have some constraints such as no short selling and no borrowing, we assume that α_t is bounded by 0 and $\alpha_{\max}$ (i.e. $0 < \alpha_t < \alpha_{\max}$), then using the same arguments as above leads to the optimal allocation:

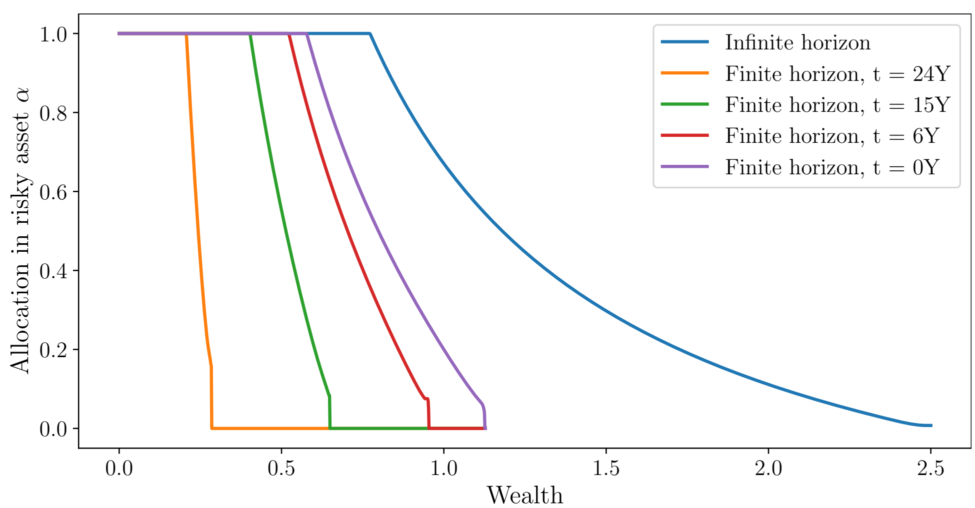
$$\alpha_t^* = \min\left\{\frac{-(\mu - r)\frac{\partial V^*}{\partial x}}{\sigma^2 X_t \frac{\partial^2 V^*}{\partial x^2}}, \alpha_{\max}\right\}$$

In this case, The HJB equation can be written as:

$$\frac{\partial V^*}{\partial t}+\min\left\{\frac{1}{2}\alpha_{\max}^2\sigma^2X_t^2\frac{\partial^2 V^*}{\partial x^2}+\alpha_{\max}(\mu-r)\frac{\partial V^*}{\partial x},-\frac{(\mu-r)^2}{2\sigma^2}\frac{\left(\frac{\partial V^*}{\partial x}\right)^2}{\frac{\partial^2 V^*}{\partial x^2}}\right\}+(rX_t-c_t)\frac{\partial V^*}{\partial x}=0 \tag{25}$$

To solve Equation 25, we use the same iterative algorithm described earlier. With an infinite investment horizon, the optimal control α_t is independent of time t and, due to time homogeneity, it is only affected by the value of portfolio wealth X_t. As a result, we have a common investment strategy for any time t. In contrast, with a finite investment horizon, the optimal control α_t is a function of time t and the portfolio wealth X_t. In other words, there are different investment strategies for different dates. As the date gets closer to the investment horizon, the threshold for ensuring on-time payments becomes smaller, and then the shape of the function of the optimal control becomes more compact, as shown in Figure 6. Therefore, this method has obvious take-profit features and involves a glide-path investment strategy in which the allocation of risky assets decreases based on the value of the portfolio.

Figure 6: Comparison of optimal control in the case of infinite and finite investment horizon (30 Y)

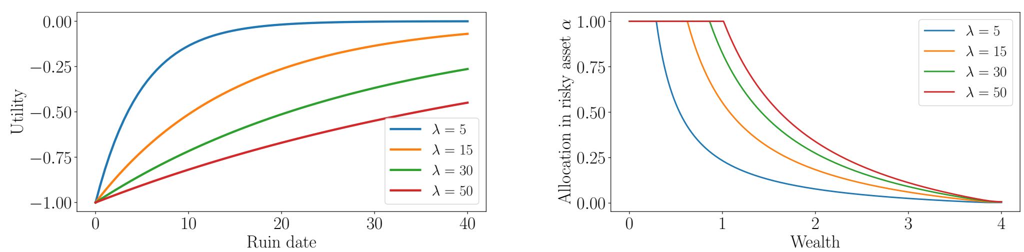


Source: Amundi Investment Institute

Impact of risk aversion parameter We simulate 1 000 scenarios for the risky asset with performance μ and volatility σ of 5% and 15% respectively, and we keep the risk-free interest rate at 1%. For simplicity, we assume that the initial wealth of the portfolio is 1. We set consumption at 4% per year with an investment horizon of 30 years. As shown in Figures 7a and 8a, when λ is small, the utility value increases rapidly according to the ruin date, which means that the investor prefers low-return and low-risk decisions, and the distribution of the total wealth is centred and the distribution of the survival time is closer to the horizon 30 years.

Figure 7: Impact of risk aversion values λ

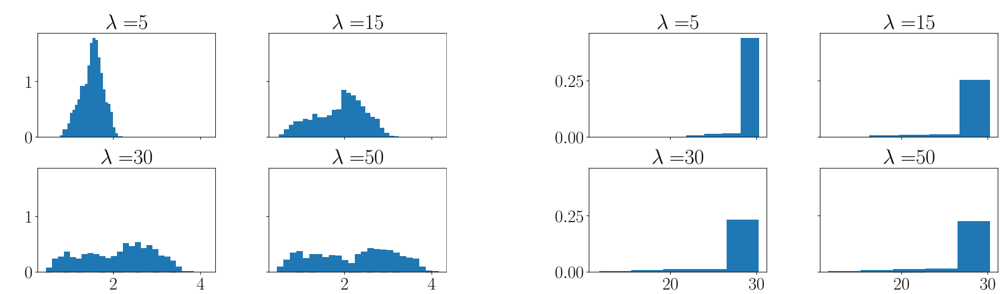
(a) Exponential utility function (b) Optimal investment strategy



Source: Amundi Investment Institute

Figure 8: Impact of risk aversion values λ (Infinite investment horizon)

(a) Total wealth (all consumption and bequest) (b) Survival time



Source: Amundi Investment Institute

Table 4: Average results of 1 000 simulations (Infinite investment horizon)

	$\lambda = 5$	$\lambda = 15$	$\lambda = 35$	$\lambda = 50$
Consumption	1.18	1.16	1.14	1.14
Wealth	1.51	1.89	2.10	2.20
Survival time (years)	29.83	29.21	28.84	28.71
Probability of sustaining 30 years	88.2%	84.7%	82.2%	80.9%
Bequest	0.33	0.73	0.96	1.06

Source: Amundi Investment Institute

Table 4 shows the average results of 1 000 simulations for the case of the infinite investment horizon. In this case, a small value for the risk aversion parameter will result in a longer average survival time and, therefore, a higher total consumption. However, this is not the case for the probability of success. When λ takes on a small value, in most scenarios, the portfolio will be conservative and run out of money as we approach the 30-year horizon because we don't take as much risk to generate performance. This is also evidenced by the fact that the smaller the risk aversion parameter, the lower the average wealth. In practice, we can set the risk aversion parameter to the value of the investment horizon.

3 Simulation

In this section, we want to assess the added value of the optimal policies compared to a standard safe withdrawal rate method, such as the well-known 4% rule. To this end, we consider three decumulation strategies:

- The standard safe withdrawal rate (SWR) method, whereby the equivalent of 4% of initial wealth is withdrawn annually. We consider 3 different constant-mix allocations between a risky asset and a risk-free asset: 50/50, 60/40, and 70/30.
- The consumption utility maximization (CUM) method, which follows the framework of Merton's portfolio problem described in Section 2.4.2. The choice of the risk aversion parameter follows Equation 16, which aims to match the different allocations used in the SWR method.
- The ruin date utility maximization (RDUM) method, which follows the framework in Section 2.4.3 to maximize the expected utility of the ruin date of the portfolio. In this strategy, we fix the consumption at the level of the standard safe withdrawal rate method, i.e., 4% of the initial wealth, and find the optimal allocation between risky and risk-free assets at each time t. We cap the allocation of risky asset $\alpha_{\max}$ at 50%, 60% and 70%.

3.1 Simulation

In this section, we model the returns of the risky and risk-free assets according to Equations 3 and 4. Then, the portfolio follows the following dynamic process:

$$\mathrm{d}X_t = \left[\left(r + \alpha_t \left(\mu - r\right)\right) X_t - c_t\right] \mathrm{d}t + \alpha_t \sigma X_t \, \mathrm{d}W_t \tag{26}$$

We set the performance μ and the volatility σ of the risky asset to 5% and 15% respectively. We consider a conservative basic scenario that sets the interest rate on the risk-free asset r at 1%. In addition, if we set the interest rate too high, which would make all decumulation strategies easy to achieve, it would be difficult to see the difference between the different methods. For simplicity, we assume that the initial wealth of the portfolio is 1 and we set the investment horizon at 30 years. The annual consumption c is fixed to 4% for the SWR method and the RDUM method. We simulate 1 000 scenarios in order to gauge the feasibility of each approach. Then, we consider two cases: a low-interest rate environment with $r = 0.5\%$ and a high annual consumption with $c = 4.5\%$. These simulations will help us better understand the characteristics of these decumulation strategies.

3.1.1 Normal conditions

In this case, we have $\mu = 5\%$, $\sigma = 15\%$, $r = 1\%$ and $c = 4\%$. We consider 3 different constant-mix allocations for the SWR method: 50/50, 60/40, and 70/30 and the simulation results are shown in Table 5. It is obvious that this type of strategy is wasteful, i.e., it has

Table 5: Simulation results of the SWR method under normal conditions

Constant-mix allocation	**50/50**	**60/40**	**70/30**
Average consumption	1.169	1.163	1.157
Average bequest	0.575	0.764	0.982
Average total wealth	1.744	1.927	2.139
Average survival time	29.274	29.124	28.966
Probability of success	83.4%	83.0%	82.3%

Source: Amundi Investment Institute

a lower probability of success but at the same time generates more returns when it survives successfully. In addition, we note that this approach generates large bequests, especially when compared to total consumption during the past 30 years. Therefore, this approach is not suitable for retirees who do not want to leave a bequest or take on too much risk.

In the CUM approach, since consumption is proportional to the wealth of the portfolio, this means that when the portfolio underperforms, we consume less and then the strategy will never default. As a result, the average survival time is 30 years and the probability of success is 100%. Therefore, the measure of the strategy can only be built around consumption, as shown in Table 6. Simulation results show clearly the instability of consumption of this method and regardless of the optimal allocation, there is always a probability of roughly 37% that the total consumption is less than the total amount of a fixed payment 4% over 30 years.

Table 6: Simulation results of the CUM method under normal conditions

γ	3.56	2.96	2.54
Optimal allocation[6]	**50/50**	**60/40**	**70/30**
Average consumption	1.661	1.655	1.650
Standard deviation consumption	0.958	0.948	0.941
10% Quantile consumption	0.782	0.784	0.785
30% Quantile consumption	1.090	1.088	1.087
Median consumption	1.408	1.404	1.401
70% Quantile consumption	1.857	1.848	1.843
90% Quantile consumption	2.911	2.883	2.863
Probability of success	100%	100%	100%
Probability that the total amount paid is less than 30×4%	37.5%	37.7%	37.6%

Source: Amundi Investment Institute

[6] The choice of the risk aversion parameter follows Equation 16, which aims to match the different allocations used in the SWR method.

Table 7 shows simulation results for the RDUM method. In this method, we have a fixed annual consumption to pay and the investment is proportional to the wealth of the portfolio. This method has a take-profit profile: the better the performance of the portfolio, the less risk we will take in the future. Compared to the SWR method shown in Table 5, the RDUM method has a higher probability of success and longer average survival time, but on average generates lower total wealth. This means that we sacrifice the possible reward for risk, i.e., bequests, in exchange for a higher success rate.

Table 7: Simulation results of the RDUM method under normal conditions

Max allocation[7]	**50/50**	**60/40**	**70/30**
Average consumption	1.193	1.194	1.194
Average bequest	0.005	0.006	0.006
Average wealth	1.199	1.200	1.200
Average survival time	29.856	29.870	29.877
Probability of success	93.4%	94.9%	95.0%

Source: Amundi Investment Institute

In conclusion, although the CUM method is based on economic theory and has an analytical solution, it does not meet the real needs of retirees, who aspire to a stable source of income after retirement. Therefore, we prefer the SWR method and the RDUM method, whose main difference lies in the balance between the total return and the probability of success of the strategy. In the following simulations, we will compare only these two strategies and examine their performance in different environments, such as low-interest rate and high consumption.

3.1.2 Low-interest rate environment

We consider a low-interest rate environment with $r = 0.5\%$. In this case, decumulation strategies are a little more difficult to succeed. The SWR method has a roughly 80% probability of success, and the RDUM method can improve this probability to 86%.

Table 8: Simulation results of the SWR method in a low-interest rate environment

Constant-mix allocation	**50/50**	**60/40**	**70/30**
Average consumption	1.162	1.157	1.153
Average bequest	0.484	0.678	0.906
Average wealth	1.646	1.835	2.059
Average survival time	29.092	28.986	28.866
Probability of success	80.1%	80.8%	80.9%

Source: Amundi Investment Institute

Figure 9 shows the quantiles of the survival time for the SWR method and the RDUM method. For a given percentage, we find that the quantile of the RDUM method is always higher than that of the SWR method, which suggests that the RDUM method outperforms the SWR method in terms of survival time, and therefore it will have a higher probability of success.

[7] We cap the allocation of risky asset $\alpha_{\max}$ at 50%, 60% and 70%.

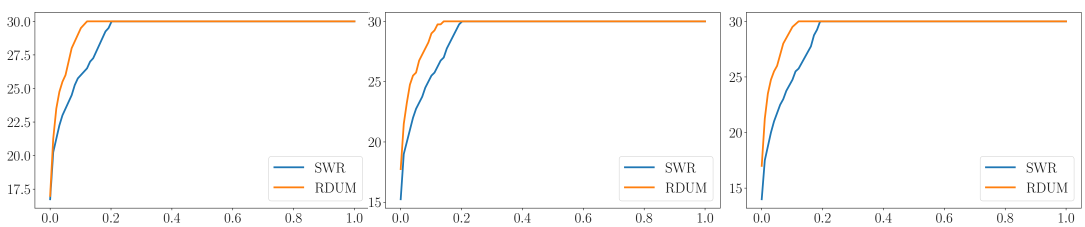
Table 9: Simulation results of the RDUM method in a low-interest rate environment

Max allocation	**50/50**	**60/40**	**70/30**
Average consumption	1.179	1.18	1.181
Average bequest	0.005	0.005	0.005
Average wealth	1.183	1.185	1.187
Average survival time	29.511	29.530	29.569
Probability of success	86.2%	86.3%	88.5%

Source: Amundi Investment Institute

Figure 9: Comparison of quantiles of the survival time for the SWR method and the RDUM method (low-interest rate environment)

(a) 50/50 (b) 60/40 (c) 70/30



Source: Amundi Investment Institute

3.1.3 High consumption

In this section, we consider an even harder situation where the fixed annual consumption is 4.5%. As shown in Tables 10 and 11, this amount of consumption will largely reduce the probability of success for both decumulation strategies. Nevertheless, the RDUM method generally outperforms the SWR method in terms of survival time.

Table 10: Simulation results of the SWR method in the case of high consumption

Constant-mix allocation	**50/50**	**60/40**	**70/30**
Average consumption	1.272	1.268	1.263
Average bequest	0.383	0.554	0.755
Average wealth	1.655	1.822	2.017
Average survival time	28.356	28.261	28.140
Probability of success	66.6%	68.6%	70.1%

Source: Amundi Investment Institute

3.2 Robustness

The main challenge in using mathematical models to solve real problems in finance lies in the gap between simple assumptions and complex reality. Therefore, the decumulation strategies we design need to be robust to assumption errors. The SWR method always uses a constant-mix investment strategy without any assumption, while the RDUM method assumes that the returns on risky assets follow a normal distribution with constant expected return and volatility over time. In this section, we simulate different scenarios where the distribution of actual returns does not follow a normal distribution and investigate the robustness of the

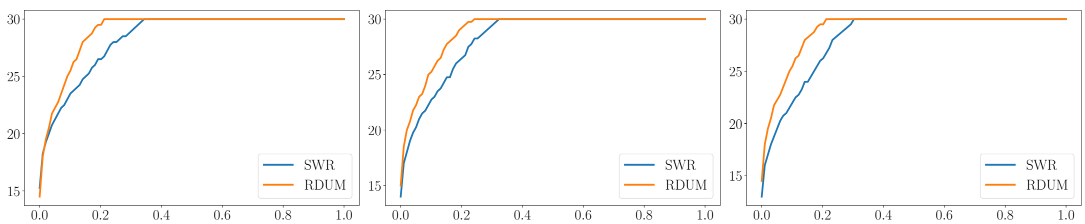
Table 11: Simulation results of the RDUM method in the case of high consumption

Max allocation	**50/50**	**60/40**	**70/30**
Average consumption	1.295	1.298	1.301
Average bequest	0.004	0.004	0.005
Average wealth	1.300	1.303	1.305
Average survival time	28.858	28.914	28.965
Probability of success	74.9%	76.6%	79.0%

Source: Amundi Investment Institute

Figure 10: Comparison of quantiles of the survival time of the SWR method and the RDUM method (high consumption)

(a) 50/50 (b) 60/40 (c) 70/30



Source: Amundi Investment Institute

RDUM method. In order to better assess the impact of the different scenarios, we set the fixed annual consumption to 4.5%, which makes it a little more difficult for the decumulation strategy to be successful over a 30-year investment horizon.

3.2.1 Bad estimation

We start with a simple case where the distribution of actual returns still follows a normal distribution, but the expected return is overestimated. The decumulation strategy is designed by using $\mu = 5\%$, while the actual returns of the risky asset are simulated with $\mu = 4\%$. Tables 12 and 13 show the simulation results for the SWR and the RDUM methods. Comparing with Tables 10 and 11 in the previous section, it is easy to see that an error in the estimation of μ leads to a reduction in the probability of success by about 10% for both methods, but the magnitude of the effect is not significantly different between them.

Table 12: Simulation results of the SWR method in the case of bad estimation

Constant-Mix allocation	**50/50**	**60/40**	**70/30**
Average consumption	1.237	1.231	1.223
Average bequest	0.250	0.360	0.485
Average wealth	1.487	1.591	1.708
Average survival time	27.602	27.462	27.282
Probability of success	54.6%	57.2%	58.6%

Source: Amundi Investment Institute

Table 13: Simulation results of the RDUM method in the case of bad estimation

Max allocation	**50/50**	**60/40**	**70/30**
Average consumption	1.267	1.272	1.275
Average bequest	0.004	0.004	0.004
Average wealth	1.270	1.276	1.279
Average survival time	28.254	28.361	28.427
Probability of success	62.8%	65.8%	69.8%

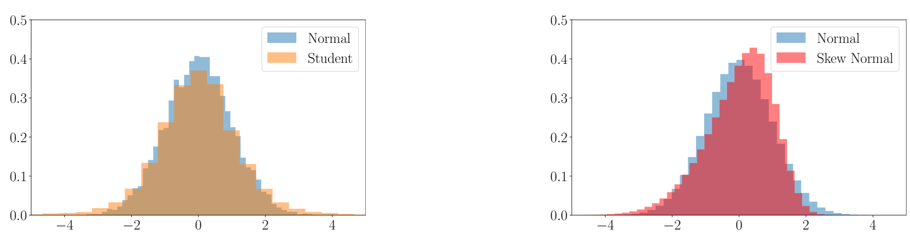
Source: Amundi Investment Institute

3.2.2 Student's t distribution

Student's t-distribution is a continuous probability distribution that generalizes the standard normal distribution, which is also symmetric around zero and bell-shaped. However, the shape of the t distribution is related to the degree of freedom ν. Compared with the standard normal distribution, the smaller the degree of freedom ν is, the flatter the t distribution curve is, i.e., the lower the middle of the curve is and the fatter the tail of the curve is, and the larger the degree of freedom ν is, the closer the t distribution is to the normal distribution. When $\nu \to +\infty$, the t distribution becomes the standard normal distribution $\mathcal{N}(0,1)$. In particular, the standard deviation of the t distribution is defined as $\frac{\nu}{\nu-2}$ and the excess kurtosis is $\frac{6}{\nu-4}$. In this section, we set the degree of freedom ν to 5 and want to use the t distribution to model the case where the volatility of the actual returns on the risky asset is greater than assumed. The first moments and the distribution are shown in Table 14 and Figure 11a. Tables 15 and 16 show the simulation results for the SWR and the RDUM methods. In the case of the SWR method, both the probability of success and the average survival time are shortened when the distribution of actual returns is more volatile and has fatter tails. In contrast, the RDUM approach only shortens the average survival time but has a higher probability of success, implying that the decumulation strategy has a better chance of being held for 30 years. This means that the distribution of survival time has higher volatility than in the normal case.

Figure 11: Normal distribution, Student's t distribution and Skew normal distribution

(a) Student's t distribution ($\nu = 5$) (b) Skew normal distribution



Source: Amundi Investment Institute

3.2.3 Skew normal distribution

In statistics, skewness corresponds to the third statistical moment of a random variable and can be used to indicate the degree of asymmetry in a probability distribution. For

Table 14: Comparison between student's t distribution and skew normal distribution

	Student's t	Skew normal
	$\nu = 5$	$\xi = 0, \omega = 1, \alpha = -3$
mean	0.00	0.00
standard deviation	1.67	1.00
skewness	0.00	-0.45
kurtosis	6.00	0.31

Source: Amundi Investment Institute

Table 15: Simulation results of the SWR method in the case of student's t distribution

Constant-mix allocation	**50/50**	**60/40**	**70/30**
Average consumption	1.258	1.251	1.242
Average bequest	0.472	0.679	0.925
Average wealth	1.729	1.930	2.167
Average survival time	28.039	27.882	27.689
Probability of success	65.2%	67.4%	67.8%

Source: Amundi Investment Institute

Table 16: Simulation results of the RDUM method in the case of student's t distribution

Max allocation	**50/50**	**60/40**	**70/30**
Average consumption	1.288	1.291	1.293
Average bequest	0.006	0.006	0.007
Average wealth	1.295	1.298	1.300
Average survival time	28.698	28.757	28.797
Probability of success	0.765	0.788	0.809

Source: Amundi Investment Institute

unimodal distributions, negative skew commonly indicates that the probability distribution plot is left-skewed, while positive skew indicates that the probability distribution plot is right-skewed. In particular, when the skewness is zero, it indicates that the data are relatively evenly distributed on both sides of the mean, not necessarily absolutely symmetrically distributed. Skewness risk in financial modeling is the risk that arises when observations are not symmetrically distributed around the mean, but rather are skewed. In addition, skewness risk usually refers specifically to the negative skewness of an asset or investment strategy, which means that there may be frequent small gains and few large losses due to stress events in finance. A negatively skewed distribution of returns is not desired by investors, as large losses tend to offset small gains. To investigate the robustness to skewness risk of the RDUM method, we assume that the actual returns of the risky asset follow a skew normal distribution[8]. To this end, we first simulate a skew normal distribution with location $\xi = 0$, scale $\omega = 1$ and shape $\alpha = -3$, the first moments and the distribution are shown in Table 14 and Figure 11b. We then scale this distribution to match the expected return $\mu = 5\%$ and the volatility $\sigma = 15\%$, and the skewness of the return distribution remains at -0.67. Tables 17 and 18 show the simulation results for the SWR and the RDUM methods, from which it can be seen that the effect of the skewed normal returns is at the same level for both methods and therefore the skewness risk for the RDUM method is not significant.

Table 17: Simulation results of the SWR method in the case of skew normal distribution

Constant-mix allocation	**50/50**	**60/40**	**70/30**
Average consumption	1.266	1.262	1.256
Average bequest	0.371	0.537	0.733
Average wealth	1.637	1.799	1.989
Average survival time	28.232	28.132	27.989
Probability of success	64.1%	66.4%	68.3%

Source: Amundi Investment Institute

Table 18: Simulation results of the RDUM method in the case of skew normal distribution

Max allocation	**50/50**	**60/40**	**70/30**
Average consumption	1.291	1.297	1.301
Average bequest	0.003	0.004	0.004
Average wealth	1.295	1.301	1.305
Average survival time	28.782	28.898	28.975
Probability of success	70.5%	74.7%	77.6%

Source: Amundi Investment Institute

In summary, we simulate three scenarios to investigate the robustness of the RDUM method compared to the SWR method: overestimation of asset returns, underestimation of volatility and fat tail, and skewness risk. In these cases, the RDUM method is affected to a similar extent as the SWR method and does not show excessive model risk.

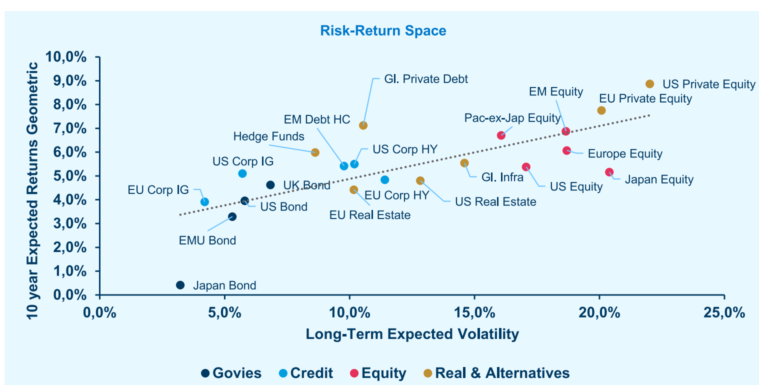
[8] More details can be found in Appendix A.10.

4 Financial application

In this section, we will test and compare the SWR method and the RDUM method with more realistic forward-looking financial data. For this purpose, we use the data simulated by Amundi Cascade Asset Simulation Model (CASM)[9]. CASM is a platform developed by Amundi in collaboration with Cambridge University. CASM combines our short-term financial and economic outlooks. It incorporates medium-term dynamics into a long-term equilibrium, to simulate forward-looking returns for different asset classes over multiple horizons.

Figure 12 shows the 10Y expected return geometric return[10] and the long-term expected volatility of each asset. In particular, expected returns are calculated using Amundi central scenario assumption, which includes the climate transition. For more detailed information, see Amundi asset class views: A rocky net zero pathway[11] (Defend *et al.*, 2023). We consider a small investment universe comprising 9 financial instruments with Equities, Government bonds, Credit Investment Grade, Credit High-yield in the US and Europe, and Emerging Market Debt. All assets are hedged to the Euro. We aim to build a well-diversified portfolio using the risk parity approach described in Appendix A.11 and this portfolio is considered as the risky asset in the SWR method and the RDUM method to design the decumulation strategy. We keep the same setting in Section 3.2 with the initial wealth of the portfolio $X_0 = 1$, the risk-free interest rate $r = 1\%$, the consumption $c = 4\%$, and the investment horizon $T = 30$ years. We also consider three constant-mix strategies for the SWR method: 50/50, 60/40, and 70/30, and these allocations are set as the upper bound for the optimal allocation in the RDUM method.

Figure 12: 10Y expected return geometric return and the long-term expected volatility of each assets[12]



Source: Amundi Asset Management CASM Model

[9]More details can be found in Appendix A.12.

[10]By definition, the arithmetic mean is always greater than or equal to the geometric mean. In particular, higher volatility of returns, higher frequency of returns, and/or a longer time horizon will increase the difference between the two measures.

[11]The paper is available at the Amundi Research Center. https://www.amundi.com/institutional/rocky-net-zero-pathway.

[12]As of July 2023. The returns are computed in local currency.

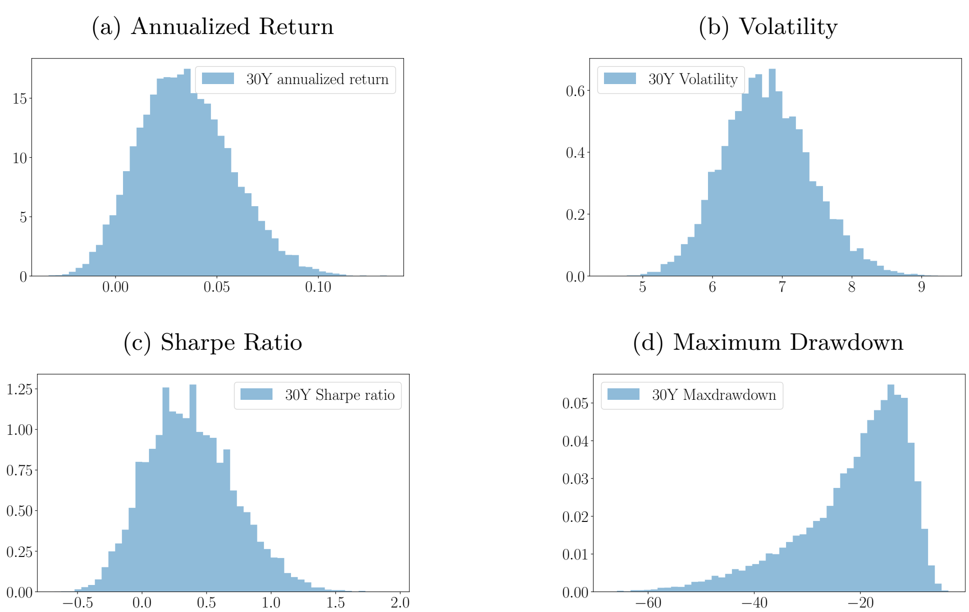
In Figure 13, we present the simulated forward-looking statistics over a 10-year horizon compared with historical statistics calculated using a 20-year sample. In the simulation of the risk parity portfolio, the length of the horizon is set to 30 years and we perform 20 000 simulations. Figure 14 and Table 19 show the distribution and quantile of simulated statistics of the risk parity portfolio. Our risk parity portfolio has different risk profile compared to the parameters used in Section 3, where the average annualized return is 3.5% and the average volatility is 6.8%.

Figure 13: Comparison of simulated forward-looking statistics and historical statistics on asset returns[13]

	Assets in local currency	SIMULATED				HISTORICAL		
		10 year Expected Returns Geometric	10 year Expected Returns Arithmetic	10 Yr Simulated Volatility	10 Yr Simulated CVaR 99%	2003-2023 Historical Returns (annualised)	2003-2023 Volatility (annualised)	2003-2023 Max Drawdown
GOVIES	US Bond	4.0%	4.0%	5.8%	13.7%	2.7%	4.8%	17.7%
	UK Bond	4.6%	4.8%	6.8%	17.6%	2.9%	7.2%	32.1%
	Japan Bond	0.4%	0.5%	3.2%	7.3%	1.3%	2.2%	9.5%
	EMU Bond All Maturity	3.3%	3.4%	5.3%	11.7%	2.7%	4.7%	20.8%
IG	Euro Corporate IG	3.9%	3.9%	4.2%	8.8%	2.7%	4.2%	16.5%
	US Corporate IG	5.1%	5.2%	5.7%	11.7%	3.9%	6.2%	20.1%
HY & EMBI	Euro Corporate HY	4.8%	5.4%	11.4%	20.7%	6.1%	10.1%	37.7%
	US Corporate HY	5.5%	5.9%	10.2%	17.1%	6.5%	9.1%	33.2%
	EM Hard Currency Debt	5.4%	5.8%	9.8%	18.4%	5.7%	8.9%	25.9%
EQUITIES	US Equity	5.4%	6.6%	17.1%	45.2%	9.4%	14.9%	51.1%
	Europe Equity	6.0%	7.5%	18.7%	49.7%	7.0%	13.9%	50.2%
	Japan Equity	5.3%	7.0%	20.4%	51.6%	6.7%	17.3%	57.4%
	Pacific ex Japan Equity	6.5%	7.7%	16.1%	44.0%	8.2%	13.7%	49.6%
	Emerging Markets Equity	7.1%	8.8%	18.7%	60.8%	9.4%	16.1%	51.9%
REAL & Alternatives	EU Real Estate	4.4%	4.8%	10.2%	33.5%			
	EU Private Equity	7.8%	9.4%	20.1%	54.3%			
	US Real Estate	4.8%	5.5%	12.8%	47.6%			
	US Private Equity	8.9%	10.8%	22.0%	53.6%			
	Global Infrastructure	5.5%	6.4%	14.6%	36.9%			
	Global Private Debt (Direct Lending)	7.1%	7.5%	10.5%	35.5%			
	Hedge Funds	6.0%	6.2%	8.6%	22.1%			

Source: Amundi Asset Management CASM Model

Figure 14: Distribution of simulated statistics of the risk parity portfolio



Source: Amundi Investment Institute

[13] As of July 2023. The returns are computed in local currency.

Table 19: Simulated statistics of the risk parity portfolio

	Annualized Return	Volatility	Sharpe Ratio	Maximum Drawdown
Mean	3.5%	6.8%	0.4	-21.2%
10% Quantile	0.6%	6.0%	-0.0	-36.1%
30% Quantile	2.1%	6.4%	0.2	-24.3%
Median	3.3%	6.8%	0.4	-18.6%
70% Quantile	4.6%	7.1%	0.5	-14.5%
90% Quantile	6.5%	7.6%	0.8	-10.6%

Table 20: Simulation results of the SWR method in the normal case

Constant-mix allocation	**50/50**	**60/40**	**70/30**
Average consumption	1.167	1.164	1.160
Standard deviation consumption	0.058	0.066	0.073
Average bequest	0.130	0.173	0.220
Average wealth	1.297	1.336	1.380
Average survival time	29.268	29.177	29.082
Probability of success	67.3%	68.2%	68.7%

Source: Amundi Investment Institute

Table 21: Simulation results of the RDUM method in the normal case

Max allocation	**50/50**	**60/40**	**70/30**
Average consumption	1.185	1.185	1.186
Standard deviation consumption	0.042	0.045	0.044
Average bequest	0.007	0.007	0.004
Average wealth	1.192	1.192	1.190
Average survival time	29.679	29.666	29.685
Probability of success	85.3%	86.0%	86.1%

Source: Amundi Investment Institute

Table 22: Simulation results of the SWR method in the low-interest rate case

Constant-mix allocation	**50/50**	**60/40**	**70/30**
Average consumption	1.149	1.151	1.151
Standard deviation consumption	0.071	0.076	0.081
Average bequest	0.088	0.133	0.187
Average wealth	1.236	1.284	1.337
Average survival time	28.84	28.87	28.866
Probability of success	54.5%	59.9%	63.2%

Source: Amundi Investment Institute

Table 23: Simulation results of the RDUM method in the low-interest rate case

Max allocation	**50/50**	**60/40**	**70/30**
Average consumption	1.155	1.159	1.163
Standard deviation consumption	0.069	0.072	0.072
Average bequest	0.003	0.005	0.006
Average wealth	1.158	1.164	1.169
Average survival time	28.987	29.065	29.148
Probability of success	61.0%	67.4%	72.5%

Source: Amundi Investment Institute

Table 24: Simulation results of the SWR method in the high consumption case

Max allocation	**50/50**	**60/40**	**70/30**
Average consumption	1.212	1.218	1.221
Standard deviation consumption	0.111	0.121	0.130
Average bequest	0.031	0.057	0.090
Average wealth	1.243	1.275	1.310
Average survival time	27.129	27.240	27.291
Probability of success	24.2%	31.0%	36.3%

Source: Amundi Investment Institute

Table 25: Simulation results of the RDUM method in the high consumption case

Constant-mix allocation	**50/50**	**60/40**	**70/30**
Average consumption	1.215	1.227	1.236
Standard deviation consumption	0.112	0.121	0.126
Average bequest	0.001	0.002	0.002
Average wealth	1.217	1.229	1.238
Average survival time	27.191	27.436	27.616
Probability of success	26.5%	35.9%	42.2%

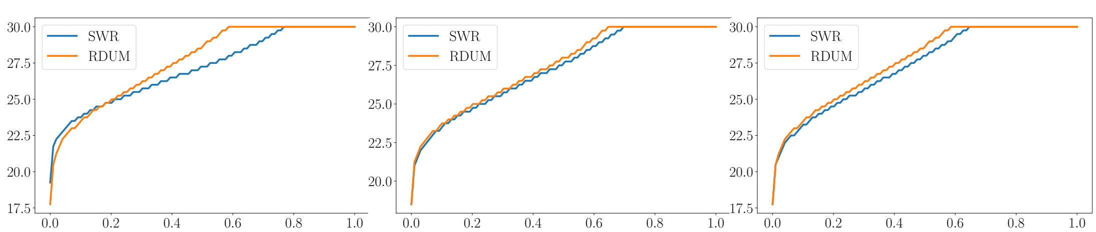
Source: Amundi Investment Institute

From Tables 20 to 25, we show the results of the SWR method and the RDUM method for three scenarios: normal conditions, low-interest rate, and high consumption, as described in Section 3.1. Under normal conditions, the results in Table 20 show that the SWR method is a wasteful strategy because we have a lower average survival time, a much lower probability of success, and more bequest than the RDUM method.

We then lower the risk-free rate, making it harder for the strategy to succeed. In this case, the RDUM method is still more effective than the SWR method from the point of view of the success rate of the strategy, but the difference is not as large as in the normal case. The third scenario is tested in a more difficult situation where consumption is set at 4.5% and the risk-free interest rate is set at 1%. In this case, the probability of success for both methods is low. If consumption levels are set too high, making it difficult for decumulation strategies to be successful, then the RDUM method converges towards the SWR method, as shown in Figure 15. Conversely, if consumption levels are appropriate, the RDUM method favors ensuring the success of the strategy rather than pursuing more gains.

Figure 15: Comparison of quantiles of the SWR method and the RDUM method

(a) 50/50 (b) 60/40 (c) 70/30



Source: Amundi Investment Institute

5 Conclusion

In this article, we discuss the general trend of pension schemes moving from DB to DC plans and the importance of having an appropriate decumulation strategy for post-retirement to avoid risks such as longevity risk, sequencing risk, inflation risk, etc. The main challenge in designing a retirement strategy is that each retiree has his or her preferences for how much to withdraw as income and how to invest savings in retirement, and these different preferences stem from different personal financial situations, health conditions, and family circumstances. As a result, it is difficult to define a common notion of "*optimal*" for each individual, meaning that there is no single solution that fits everyone.

In this paper, we detail three methods used to design decumulation strategies: the standard safe withdrawal rate (SWR) method, the consumption utility maximization (CUM) method, and the ruin date utility maximization (RDUM) method. We test the feasibility and the robustness of these methods with simulated data based on geometric Brownian motion and certain scenarios such as overestimation of asset returns, underestimation of volatility and fat tail, and skewness risk.

The SWR method allows retirees to regularly withdraw a fixed amount of money to meet their needs and chooses a constant-mix or a glide-path investment strategy to invest the remaining wealth in their portfolios. One of the famous strategies following the SWR method is the 4% rule, which pays a constant real income corresponding to the 4% of initial wealth at retirement and chooses a constant-mix allocation between stocks and bonds. This strategy is simple and comprehensible for retirees, however, it is considered a wasteful strategy, as in many scenarios, there is an excess of assets at the end of 30 years. The CUM method is based on economic theory and seeks the optimal consumption and investment strategy for the remaining funds the portfolio at each date, by maximizing the expected discounted utility of lifetime consumption. Using this method, the optimal consumption is not constant but varies with the value of the portfolio. While the CUM method will never run out of money, it is still not suitable for retirees because it does not produce a steady income.

The RDUM method, like the CUM method, also follows stochastic optimal control theory, and its objective is to maximize the expected utility of the ruin date. In this method, the consumption is fixed or programmed for each date, and the problem is finding an optimal investment strategy. Although it is often difficult to find an analytical solution for this method, numerical solutions to the HJB partial differential equations can be obtained by the finite difference method. This method also involves a glide-path investment strategy, but the allocation of risky assets decreases with the value of the portfolio rather than over time, which makes more sense than the SWR method. While there are many metrics to compare different decumulation strategies, such as the average total withdrawal during the whole period of retirement, the average survival time, the probability of success, etc., we believe the most important is the probability of success. Strategies should be set with clear consumption goals and a fixed investment horizon, and be designed to maximize the probability of success rather than other metrics. Because the real world is only a sample in the simulation, but this piece of data is related to all retirees, that is, if the future corresponds to a bad scenario, then all retirees using a similar strategy will experience the same difficulty of running out of money. It is therefore important to maximize the probability of success of the strategy. Compared to the SWR method, the RDUM method has a take-profit feature and thus a higher probability of success. We find that the strategy based on the RDUM method is also robust in cases of overestimated return of risky assets, thick tails, and negative skewness. In practical application, this method is customizable, allowing retirees to choose different allocation constraints and consumption patterns, such as constant, inflation-resistant, and U-shaped, depending on their actual situation. This approach is therefore suitable for retirees with clear goals, certainty of consumption, and relative conservatism. In addition, those who wish to avoid longevity risk, can set a longer investment horizon, say 40 years. Another way to address longevity risk is to incorporate an annuity or a modern tontine into a decumulation strategy. It can be viewed as an exit strategy from the drawdown strategies described in this paper, or it can be considered directly as an asset in the portfolio. However, the act of purchasing an annuity or modern pension is irreversible, so timing is critical.

There are several ways to improve our work. In further research, we could consider incorporating transaction costs to study their impact or approximating the solution of the RDUM method with a stepwise function to reduce transaction costs. Another interesting study would be to consider more complex financial market models, such as jump models, or the use of machine learning techniques such as Restricted Boltzmann Machine (RBM) and Generative Adversarial Networks (GAN) (Lezmi *et al.*, 2020). We can then use these models to simulate more realistic financial data to investigate the robustness of decumulation strategies.

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A Appendix

A.1 Abbreviations

- CARA: Constant absolute risk aversion
- CASM: Cascade asset simulation model
- CRRA: Constant relative risk aversion
- CUM: Consumption utility maximization
- DB: Defined-benefit
- DC: Defined-contribution
- DPP: Dynamic programming principle
- FDM: Finite difference methods
- GAN: Generative adversarial network
- HJB: Hamilton-Jacobi-Bellman
- i.i.d: Independent and identically distributed
- ODE: Ordinary differential equation
- PAYG: Pay-as-you-go
- PDE: Partial differential equation
- PIDE: Partial integro-differential equation
- NAV: Net asset value
- RDUM: Ruin date utility maximization
- SDE: Stochastic differential equation
- SN: Skew normal
- SWR: Safe withdrawal rate
- RBM: Restricted Boltzmann machine

A.2 Annuitization

In most countries, Social Security usually provides the most basic livelihood protection, so retirees often purchase annuity products to protect their retirement. An annuity is a contract between an individual and an insurance company whereby the retiree pays a lump sum of money on or after the date of retirement in order to have the right to receive regular payments for the rest of his life or for a specified period (e.g., from age 80 to 95). The payments are often fixed but can also be inflation-indexed or completely variable, in which case they are linked to the value of a chosen investment portfolio. In addition, the payments can be either immediate, i.e., regular payments begin when the annuity is purchased, or deferred, in which case the retiree will receive regular payments after a specified date (e.g., after age 80).

In the case of a life annuity, when regular payments of a purchased annuity are to continue until the death of the annuitant, the longevity risk of retirees is transferred to the insurer's default risk. In addition, unless there are other beneficiaries in the contract, the annuity does not leave a bequest for the annuitant's relatives. Thus, as explained in Yaari (1965), annuities are an appropriate decumulation strategy for retirees with more longevity risk aversion and no bequest motive.

However, purchasing an annuity is not a simple decision. The longevity risk for retirees and the default risk for insurance companies are not easy to assess. The challenge for a life annuity is the timing of the purchase, which is a very personalized issue as it depends greatly on an individual's health and financial goals. As recommended by Fullmer (2009), we can use life annuities as the benchmark for a decumulation strategy, i.e., if the value of the retiree's pension fund is lower than the cost of annuitizing his or her required income stream, a life annuity should be purchased rather than continuing with a decumulation strategy.

A.3 Tontine

The term "*Tontine*" derives from the name of Lorenzo de Tonti, a Neapolitan banker and governor of the Italian province of Gaeta, who served as a financial advisor to the French Crown in the 1850s and proposed the original tontine program to Cardinal Mazarin to raise funds for the military expenditure of King Louis XIV. Unlike ordinary annuity plans, all investors in the same tontine are treated as a group, and tontine's dividends are proportional to the number of surviving investors in the tontine. When each investor dies, they are not replaced by a new investor, but their share is divided equally among the surviving investors. Surviving investors therefore profit from the deaths of others, and the longer the investors live, the more dividends they receive. This process continues until the last investor dies and the organizers of tontine absorb the remaining capital.

Lorenzo de Tonti's proposal was rejected by the Paris Parliament. It was not until 1670 that the first tontine was organized in the Dutch city of Kampen, and it was not long before other European countries followed suit, such as France in 1689 and England in 1693. After this, tontine continued to grow as it met both the financing needs of the government and the retirement needs of the population, both of which contributed to the rapid spread of tontine schemes in the 17th and 18th centuries. Particularly in the mid-19th century, American insurance companies began to issue tontine insurance to the public, and the tontine scheme entered a period of super-rapid growth. However, shortly after 1900, several sensational insurance embezzlement scandals led to an investigation of tontine insurance by the Armstrong Committee[14], and tontine was eventually banned in the United States. Reasons for the ban included the gambling nature of the product, false advertising in the actual sale, lack of supervision in the management of insurance companies leading to corruption, falsification of accounts, and misappropriation of public funds.

The advantage of tontine is that it can offer a solution to longevity risk: the longer the life span, the greater the benefit received. If properly regulated, tontine has the potential to be a suitable insurance product that meets people's retirement needs. Today, tontines are regulated in Europe by Directive 2002/83/EC of the European Parliament and of the Council. Many authors proposed a modern version of the tontine to help people finance their final years, such as Piggott *et al.* (2005), Stamos (2008), Donnelly *et al.* (2014) and

[14] Joint Committee of the Senate and Assembly of the State of New York to Investigate and Examine into the Business and Affairs of Life Insurance Companies Doing Business in the State of New York.

Milevsky and Salisbury (2015). As reported in the New York Times in March 2017, tontines are getting new consideration as a way for people to get a stable retirement income.

A.4 HJB equation for decumulation strategies by maximizing the expected discounted utility of lifetime consumption

We define the value function $V(t, X_t)$ at time t:

$$\begin{aligned} V(t, X_t) &= \mathbb{E}\left[\int_t^T e^{-\rho(s-t)} F(s, c_s)\, \mathrm{d}s + e^{-\rho(T-t)} G(X_T) \,|X_t\right] \\ &= \mathbb{E}_t\left[\int_t^T e^{-\rho(s-t)} F(s, c_s)\, \mathrm{d}s + e^{-\rho(T-t)} G(X_T)\right] \end{aligned}$$

Thus, the optimal value function $V^\star(t, X_t)$ is:

$$V^*(t, X_t) = \max_{c_t, \alpha_t} V(t, X_t)$$

By applying the dynamic programming principle, we obtain:

$$\begin{aligned} V^*(t, X_t) &= \max_{c_t, \alpha_t} V(t, X_t) \\ &= \max_{c_t, \alpha_t} \mathbb{E}_t\left[\int_t^T e^{-\rho(s-t)} F(s, c_s)\, \mathrm{d}s + e^{-\rho(T-t)} G(X_T)\right] \\ &= \max_{c_t, \alpha_t} \mathbb{E}_t\left[\int_t^{t+h} e^{-\rho(s-t)} F(s, c_s)\, \mathrm{d}s + \int_{t+h}^T e^{-\rho(s-t)} F(s, c_s)\, \mathrm{d}s + e^{-\rho(T-t)} G(X_T)\right] \\ &= \max_{c_t, \alpha_t} \mathbb{E}_t\left[\int_t^{t+h} e^{-\rho(s-t)} F(s, c_s)\, \mathrm{d}s + e^{-\rho h}\mathbb{E}_{t+h}\left[\int_{t+h}^T e^{-\rho(s-t-h)} F(s, c_s)\, \mathrm{d}s + e^{-\rho(s-t-h)} G(X_T)\right]\right] \\ &= \max_{c_t, \alpha_t} \mathbb{E}_t\left[\int_t^{t+h} e^{-\rho(s-t)} F(s, c_s)\, \mathrm{d}s + e^{-\rho h} V^*(t+h, X_{t+h})\right] \end{aligned} \tag{27}$$

Let $\tilde{V}^*(t, X_t) = e^{-\rho t} V^*(t, X_t)$ and we have

$$\tilde{V}^*(t, X_t) = \max_{c_t, \alpha_t} \mathbb{E}_t\left[\int_t^{t+h} e^{-\rho s} F(s, c_s)\, \mathrm{d}s + \tilde{V}^*(t+h, X_{t+h})\right] \tag{28}$$

According to the Itô formula, we obtain:

$$\begin{aligned} \tilde{V}^*(t+h, X_{t+h}) = \tilde{V}^*(t, X_t) \\ &+ \int_t^{t+h} \left\{\frac{\partial \tilde{V}^*}{\partial t}(s, X_s) + \tilde{\mu}\frac{\partial \tilde{V}^*}{\partial x}(s, X_s) + \frac{1}{2}\tilde{\sigma}^2\frac{\partial^2 \tilde{V}^*}{\partial x^2}(s, X_s)\right\} \mathrm{d}s \\ &+ \int_t^{t+h} \tilde{\sigma}\frac{\partial \tilde{V}^*}{\partial x}(s, X_s)\, \mathrm{d}W_s \end{aligned} \tag{29}$$

In particular,

$$\frac{\partial \tilde{V}^*}{\partial t}(s, X_s) = -\rho e^{-\rho s} V^*(t, X_s) + e^{-\rho t} \frac{\partial V^*}{\partial t}(s, X_s)$$

$$\frac{\partial \tilde{V}^*}{\partial x}(s, X_s) = e^{-\rho s} \frac{\partial V^*}{\partial x}(s, X_s)$$

$$\frac{\partial^2 \tilde{V}^*}{\partial x^2}(s, X_s) = e^{-\rho s} \frac{\partial^2 V^*}{\partial x^2}(s, X_s)$$

Then, we combine Equations 28 and 29:

$$\begin{aligned}
\max_{c_t, \alpha_t} \mathbb{E}_t \Bigg[& \int_t^{t+h} e^{-\rho s} F(s, c_s) \, \mathrm{d}s \\
& + \int_t^{t+h} \left\{ \frac{\partial \tilde{V}^*}{\partial t}(s, X_s) + \tilde{\mu} \frac{\partial \tilde{V}^*}{\partial x}(s, X_s) + \frac{1}{2} \tilde{\sigma}^2 \frac{\partial^2 \tilde{V}^*}{\partial x^2}(s, X_s) \right\} \mathrm{d}s \\
& + \int_t^{t+h} \tilde{\sigma} \frac{\partial \tilde{V}^*}{\partial x}(s, X_s) \, \mathrm{d}W_s \Bigg] = 0
\end{aligned} \tag{30}$$

Let h tend to zero, we can simplify the equation as:

$$\max_{c_t, \alpha_t} \left[e^{-\rho t} F(t, c_t) + \frac{\partial \tilde{V}^*}{\partial t}(t, X_t) + \tilde{\mu} \frac{\partial \tilde{V}^*}{\partial x}(t, X_t) + \frac{1}{2} \tilde{\sigma}^2 \frac{\partial^2 \tilde{V}^*}{\partial x^2}(t, X_t) \right] = 0$$

For simplicity, we omit (t, X_t) for $\tilde{V}^*(t, X_t)$ and $V^*(t, X_t)$. Then, we have:

$$\max_{c_t, \alpha_t} \left[e^{-\rho t} F(t, c_t) + -\rho e^{-\rho t} V^* + e^{-\rho t} \frac{\partial V^*}{\partial t} + \tilde{\mu} e^{-\rho t} \frac{\partial V^*}{\partial x} + \frac{1}{2} \tilde{\sigma}^2 e^{-\rho t} \frac{\partial^2 V^*}{\partial x^2} \right] = 0$$

Finally, since ρV^* and $\frac{\partial V^*}{\partial t}$ are independent of c_t, α_t, we can rewrite the above equation as:

$$\frac{\partial V^*}{\partial t} + \max_{c_t, \alpha_t} \left[F(t, c_t) + \tilde{\mu} \frac{\partial V^*}{\partial x} + \frac{1}{2} \tilde{\sigma}^2 \frac{\partial^2 V^*}{\partial x^2} \right] = \rho V^* \tag{31}$$

A.5 Analytical solution for Merton's portfolio problem

According to Rao and Jelvis (2022), we can surmise with a guess solution in terms of a deterministic function f of time:

$$V^*(t, X_t) = f(t)^\gamma \frac{X_t^{1-\gamma}}{1-\gamma}$$

We have

$$\frac{\partial V^*}{\partial t} = \gamma f(t)^{\gamma-1} f'(t) \frac{X_t^{1-\gamma}}{1-\gamma}$$

$$\frac{\partial V^*}{\partial x} = f(t)^\gamma X_t^{-\gamma}$$

$$\frac{\partial^2 V^*}{\partial x^2} = -\gamma f(t)^\gamma X_t^{-\gamma-1}$$

Substituting the guess solution into Equation 14, we obtain:

$$f'(t) = \frac{\rho - (1-\gamma) \cdot \left(\frac{(\mu - r)^2}{2\sigma^2 \gamma} + r \right)}{\gamma} f(t) - 1$$

Then, we have

$$f(t) = \begin{cases} \frac{1 + (\nu\epsilon - 1) e^{-\nu(T-t)}}{\nu} & \text{for } \nu \neq 0 \\ T - t + \epsilon & \text{for } \nu = 0 \end{cases}$$

where

$$\nu = \frac{\rho - (1-\gamma) \cdot \left(\frac{(\mu - r)^2}{2\sigma^2 \gamma} + r \right)}{\gamma}$$

Finally, we obtain:

$$\begin{aligned} \alpha_t^* &= \frac{-(\mu - r) \frac{\partial V^*}{\partial x}}{\sigma^2 X_t \frac{\partial^2 V^*}{\partial x^2}} \\ &= \frac{(\mu - r) f(t)^\gamma X_t^{-\gamma}}{\sigma^2 X_t \gamma f(t)^\gamma X_t^{-\gamma - 1}} \\ &= \frac{\mu - r}{\sigma^2 \gamma} \end{aligned} \tag{32}$$

$$\begin{aligned} c_t^* &= \left(\frac{\partial V^*}{\partial x} \right)^{-\frac{1}{\gamma}} \\ &= \frac{X_t}{f(t)} \\ &= \begin{cases} \frac{\nu X_t}{1 + (\nu\epsilon - 1) e^{-\nu(T-t)}} & \text{for } \nu \neq 0 \\ \frac{X_t}{T - t + \epsilon} & \text{for } \nu = 0 \end{cases} \end{aligned} \tag{33}$$

A.6 HJB equation for decumulation strategies by maximizing the expected utility of the ruin date

By applying the dynamic programming principle, we can obtain the following equation:

$$\begin{aligned} V^*(t, X_t) &= \max_{\alpha_t} V(t, X_t) \\ &= \max_{\alpha_t} \mathbb{E}_t \left[\mathbb{1}_{\tau_h < t+h} U(\tau_h) + \mathbb{1}_{\tau_h \geq t+h} V^*(t+h, X_{t+h}) \right] \end{aligned} \tag{34}$$

According to Itô formula, we obtain:

$$\begin{aligned} V^*(t+h, X_{t+h}) = V^*(t, X_t) & \\ + \int_t^{t+h} & \left\{ \frac{\partial V^*}{\partial t}(s, X_t) + \tilde{\mu} \frac{\partial V^*}{\partial x}(s, X_t) + \frac{1}{2} \tilde{\sigma}^2 \frac{\partial^2 V^*}{\partial x^2}(s, X_t) \right\} \mathrm{d}s \\ + \int_t^{t+h} & \tilde{\sigma} \frac{\partial V^*}{\partial x}(s, X_t) \, \mathrm{d}W_s \end{aligned} \tag{35}$$

where $\tilde{\mu} = [(r + (\mu - r)\,\alpha_t)\,X_t - \bar{c}_t]$ and $\tilde{\sigma} = \sigma\alpha_t X_t$. Let h tend to zero, we can simplify the equation as:

$$\max_{\alpha_t}\left[\frac{\partial V^*}{\partial t} + \tilde{\mu}\frac{\partial V^*}{\partial x} + \frac{1}{2}\tilde{\sigma}^2\frac{\partial^2 V^*}{\partial x^2}\right] = 0$$

As $\frac{\partial V^*}{\partial t}$ is independent of α_t, we have:

$$\frac{\partial V^*}{\partial t} + \max_{\alpha_t}\left[\tilde{\mu}\frac{\partial V^*}{\partial x} + \frac{1}{2}\tilde{\sigma}^2\frac{\partial^2 V^*}{\partial x^2}\right] = 0$$

A.7 Analytical solution to the ruin date problem in the case of infinite investment horizon

In the case of infinite investment horizon, the HJB equation associated with the ruin date problem can be reduced to an ordinary differential equation:

$$-\frac{1}{\lambda}V^* - \frac{(\mu - r)^2}{2\sigma^2}\frac{\left(\frac{\partial V^*}{\partial x}\right)^2}{\frac{\partial^2 V^*}{\partial x^2}} + (rx - \bar{c}_t)\,\frac{\partial V^*}{\partial x} = 0 \tag{36}$$

Let $s = -\frac{(\mu - r)^2}{2\sigma^2}$, then:

$$\lambda^{-1}\frac{V^*}{\frac{\partial V^*}{\partial x}} = s\frac{\frac{\partial V^*}{\partial x}}{\frac{\partial^2 V^*}{\partial x^2}} + (rx - \bar{c}_t) \tag{37}$$

We can try to solve Equation 36 with a guess solution of the form $V^*(t, x) = (rxk - \bar{c}_t k)^p$. As a result, we have:

$$\begin{aligned}\frac{\partial V^*}{\partial x} &= rkp\,(rxk - \bar{c}_t k)^{p-1}\\ \frac{\partial^2 V^*}{\partial x^2} &= (rk)^2\,p\,(p-1)\,(rxk - \bar{c}_t k)^{p-2}\end{aligned} \tag{38}$$

Hence,

$$\lambda r p^2 - [\lambda(r - s) + 1]p + 1 = 0 \tag{39}$$

Consider Equation 39, it has 2 roots due to:

$$\Delta = [\lambda(r - s) + 1]^2 - 4\lambda r \geq (\lambda r + 1)^2 - 4\lambda r = (\lambda r - 1)^2 \geq 0 \tag{40}$$

where $-s = \frac{(\mu - r)^2}{2\sigma^2} \geq 0$. Denote the greater root of 39:

$$p = \frac{[\lambda(r - s) + 1] + \sqrt{\Delta_a}}{2\lambda r} \geq \frac{-s\lambda + \lambda r + 1 + |\lambda r - 1|}{2\lambda r} \geq \frac{\tilde{\mu}^2}{4r\sigma^2} + 1 \geq 1 \tag{41}$$

In case of vanishing risk aversion $\lambda \to \infty$:

$$p = \frac{(r - s) + \frac{1}{\lambda} + \frac{\Delta_a}{\lambda^2}}{2r} = \frac{(r - s) + \frac{1}{\lambda} + \sqrt{[r - s + \frac{1}{\lambda}]^2 - \frac{4r}{\lambda}}}{2r} \to \frac{r - s}{r} = 1 + \frac{\tilde{\mu}^2}{2r\sigma^2} \tag{42}$$

Now, we have $u^* = [k(c - vr)]_a^p$. The identification of the solution is done with the boundary conditions.

Boundary conditions and optimal solution

- First boundary condition: It is observed that when the retiree withdraws a constant amount c, the expression $\epsilon(x) = c - xr \leq 0$ holds $\forall x \geq x_{\max} = c/r$. In other words, if the wealth value x reaches the value of $x_{\max}$, the interest from the bank account is sufficient to cover the coupon payment, rendering the problem trivial and resulting in $u(x) = 0 \ \forall x \geq x_{\max}$. Consequently, our analysis will focus solely on the range $x_t \in [x_{thres}, x_{\max}]$. Therefore, the first boundary condition can be expressed as $u(x_{max}) = 0$.

- Second boundary condition: $u(x_{thres}) = -1$

From the second boundary condition, we can find that $k = 1/(x_{thres}r - c)$. As a result, we have:

$$\begin{aligned} u^*(x) &= -\left[\frac{c - xr}{c - x_{thres}r}\right]^p \\ \alpha^*(x) &= \frac{\tilde{\mu}}{\sigma^2(p-1)} \frac{c - xr}{xr} = \kappa_a \frac{x_{\max} - x}{x} \end{aligned} \tag{43}$$

In case of vanishing risk aversion $\lambda \to \infty$:

$$\alpha^*(x, \lambda \to \infty) = \frac{2r}{\tilde{\mu}} \frac{c - xr}{xr} = \frac{2r}{\tilde{\mu}} \frac{x_{\max} - x}{x} \tag{44}$$

The optimal wealth path follows a hyperbolic shape, representing the trade-off between wealth accumulation and risk exposure. Consequently, the adjustment of risky exposure occurs at a pace greater than a linear one. As we approach the target capital, the need for a high-risk asset shrinks as it generates sufficient investment income. Consequently, the risky exposure can be reduced. This hyperbolic strategy effectively avoids drawdown issues as wealth increases, since the risky exposure diminishes rapidly. Conversely, the strategy exhibits mean-reverting behavior, wherein during bad market conditions, i.e., wealth decreasing, the risky exposure is increased in anticipation of future market rebound. This behavior stems from the assumption that the expected return, μ, is positive. The ratio $\frac{(x_{\max} - x)}{x}$ can be interpreted as a "*gap leverage*", a required leverage to close out the gap between the current wealth X and the required weight $x_{\max}$. We may approximate a finite horizon problem with an explicit solution in plugging naively $x_{\max}(t) = c/r\,(1 - \exp\ (-r(T-t)))$ In addition, κ can be interpreted as a base exposure which depends on the support of the current investment environment and retiree's preference since it depends only on r, μ, σ, ρ but not on c.

A.8 Fully implicit scheme for the time-dependant problem

Suppose α^* is given, and that we aim to solve:

$$\frac{\partial V^*}{\partial t} + (r + (\mu - r)\,\alpha^* X_t - \bar{c}_t)\,\frac{\partial V^*}{\partial x} + \frac{1}{2}{\alpha^*}^2\sigma^2 X_t^2 \frac{\partial^2 V^*}{\partial x^2} = 0 \tag{45}$$

Or, using a centered differentiation scheme:

$$\frac{V^*_{t+\mathrm{d}t,i} - V^*_{t,i}}{\mathrm{d}t} + (r + (\mu - r)\,{\alpha^*}_i X_t - \bar{c}_t)\,\frac{V^*_{t,i+1} - V^*_{t,i-1}}{2\mathrm{d}x} + \frac{1}{2}{\alpha_i^*}^2\sigma^2 X_t^2 \frac{V^*_{t,i+1} + V^*_{t,i-1} - 2V^*_{t,i}}{(\mathrm{d}x)^2} = 0 \tag{46}$$

In fact, the centered scheme is not the most stable. We can introduce a forward/backward scheme for the first derivative, depending on the sign of the first derivative coefficient:

$$\frac{V^*_{t+\mathrm{d}t,i} - V^*_{t,i}}{\mathrm{d}t} + \hat{\mu}\left(\frac{V^*_{i+1} - V^*_{i}}{\mathrm{d}x} 1_{\{\hat{\mu}>0\}} + \frac{V^*_{i} - V^*_{i-1}}{\mathrm{d}x} 1_{\{\hat{\mu}<0\}}\right) + \frac{1}{2}{\alpha_i^*}^2\sigma^2 X_t^2 \frac{V^*_{i+1} + V^*_{i-1} - 2V^*_{i}}{(\mathrm{d}x)^2} = 0$$

where $\hat{\mu} = (r + (\mu - r)\,{\alpha^*}_i X_t - \bar{c}_t)$

A.9 Fully implicit scheme for the stationary problem

Suppose α^* is given, and that we aim to solve:

$$\frac{-1}{\lambda} V^* + (r + (\mu - r)\,\alpha^* X_t - \bar{c}_t)\,\frac{\partial V^*}{\partial x} + \frac{1}{2}{\alpha^*}^2\sigma^2 X_t^2 \frac{\partial^2 V^*}{\partial x^2} = 0 \tag{47}$$

Or, using a centered differentiation scheme:

$$-\frac{1}{\lambda} V_i^* + (r + (\mu - r)\,{\alpha^*}_i X_t - \bar{c}_t)\,\frac{V^*_{t,i+1} - V^*_{t,i-1}}{2\mathrm{d}x} + \frac{1}{2}{\alpha_i^*}^2\sigma^2 X_t^2 \frac{V^*_{t,i+1} + V^*_{t,i-1} - 2V^*_{t,i}}{(\mathrm{d}x)^2} = 0 \tag{48}$$

In fact, the centered scheme is not the most stable. We can introduce a forward/backward scheme for the first derivative, depending of the sign of the first derivative coefficient:

$$-\frac{1}{\lambda} V_i^* + \hat{\mu}\left(\frac{V^*_{i+1} - V^*_{i}}{\mathrm{d}x} 1_{\{\hat{\mu}>0\}} + \frac{V^*_{i} - V^*_{i-1}}{\mathrm{d}x} 1_{\{\hat{\mu}<0\}}\right) + \frac{1}{2}{\alpha_i^*}^2\sigma^2 X_t^2 \frac{V^*_{i+1} + V^*_{i-1} - 2V^*_{i}}{(\mathrm{d}x)^2} = 0$$

where $\hat{\mu} = (r + (\mu - r)\,{\alpha^*}_i X_t - \bar{c}_t)$

A.10 The skew normal distribution

The skew normal (SN) distribution is a continuous probability distribution that generalises the normal distribution to allow for non-zero skewness, which was first introduced by O'Hagan and Leonard (1976). We adopt here the construction of Azzalini and Dalla Valle (1996) and Azzalini and Capitanio (1999).

The probability density function of a skew normal distribution with location ξ, scale ω

and shape α is defined as follows:

$$f(x) = \frac{2}{\omega}\phi\left(\frac{x-\xi}{\omega}\right)\Phi\left(\alpha\left(\frac{x-\xi}{\omega}\right)\right)$$

where $\phi(x)$ and $\Phi(x)$ denote the probability density function and the cumulative distribution function of a standard normal distribution $\mathcal{N}(0,1)$, respectively. The first four moments of the univariate SN distribution X are:

$$\begin{aligned}
\mu(X) &= \xi + \omega\eta\sqrt{\frac{2}{\pi}} \\
\sigma^2(X) &= \omega^2\left(1 - \frac{2\eta^2}{\pi}\right) \\
\text{skewness}(X) &= \frac{4-\pi}{2}\frac{(\eta\sqrt{2/\pi})^3}{(1-2\eta^2/\pi)^{3/2}} \\
\text{excess kurtosis}(X) &= 2(\pi-3)\frac{(\eta\sqrt{2/\pi})^4}{(1-2\eta^2/\pi)^2}
\end{aligned}$$

where $\eta = \frac{\alpha}{\sqrt{1+\alpha^2}}$.

A.11 Risk budgeting approach

The fundamental principle of the risk budgeting (RB) approach is to allocate funds based on risk, rather than capital, as stated in Roncalli (2013). To achieve this, we introduce the concept of risk contribution, which is characterized as the contribution of each asset in the portfolio to the portfolio's overall risk. The portfolio manager defines a set of risk budgets and then determines the weights of the portfolio such that the risk contributions are in line with the budgets.

From a mathematical point of view, a risk budgeting portfolio is defined as follows:

$$\begin{cases}
\mathcal{RC}_i(x) = b_i\mathcal{R}(x) & \\
b_i > 0, x_i \geq 0 & \text{for all } i \\
\sum_{i=1}^n b_i = 1, \sum_{i=1}^n x_i = 1 &
\end{cases} \tag{49}$$

where x_i is the allocation of Asset i, $\mathcal{R}(x)$ is the risk of the portfolio, which is typically the volatility of the portfolio, $\mathcal{RC}_i(x)$ and b_i are respectively the risk contribution and the risk budget of Asset i.

A route to solving Problem (49) is to transform the non-linear system into an optimization problem:

$$\begin{aligned}
x^\star = \arg\min \sum_{i=1}^n & \left(\mathcal{RC}_i(x) - b_i\mathcal{R}(x)\right)^2 \\
\text{s.t.} \quad & x_i \geq 0, b_i \geq 0 \quad \text{for all } i \\
& 1^\top x = 1 \\
& 1^\top b = 1
\end{aligned} \tag{50}$$

However, Problem (50) is not a convex problem (Feng and Palomar, 2015), and the optimization has some numerical issues, particularly in the high-dimensional case, that is, when the number of assets is large. Roncalli (2013) shows a different approach to solving Problem (49) with the help of the logarithmic barrier method and the solution is:

$$x_{\mathrm{RB}} = \frac{y^\star}{\mathbf{1}^\top y^\star}$$

where $y^\star(c)$ is the solution of the alternative optimization problem:

$$\begin{aligned} y^\star(c) &= \arg\min \mathcal{R}(y) \\ \text{s.t.} \quad & \sum_{i=1}^{n} b_i \ln y_i \geq c \\ & y_i \geq 0 \quad \text{for all } i \end{aligned}$$

where c is an arbitrary scalar. This problem can be solved by the Newton algorithm (Chaves *et al.*, 2012) and the Cyclical Coordinate Descent (CCD) algorithm (Griveau-Billion *et al.*, 2013). In the special case of the equal risk contribution (ERC) portfolio, i.e. the risk budgets are the same ($b_i = b_j, \text{ for all } i, j$), we have:

$$\begin{aligned} y^\star(\lambda) &= \arg\min \mathcal{R}(y) \\ \text{s.t.} \quad & \sum_{i=1}^{n} \ln y_i \geq c \\ & y_i \geq 0 \quad \text{for all } i \end{aligned}$$

Finally,

$$x_{\mathrm{ERC}} = \frac{y^\star}{\mathbf{1}^\top y^\star}$$

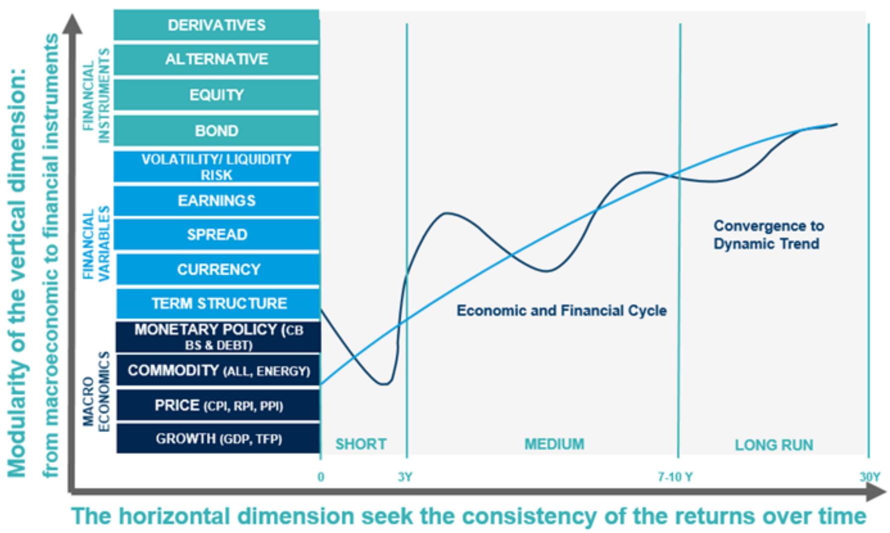
This portfolio allocation strategy is also known as the risk parity approach, which is the main alternative method to the traditional mean-variance portfolio optimization.

A.12 Cascade Asset Simulation Model

Reasonable investment solutions require a transparent and comprehensive view of the capital markets. This is especially true for institutional investors such as pension funds or endowments with medium to long-term horizons in need of a coherent strategy to maximize the probability of achieving their stated objectives. For such tasks, it is imperative to simulate scenarios projecting possible trends, in not just the prices of instruments but additionally, in the underlying risk factors and the their complex interactions. The process for generating the scenarios reflecting our view of economic and financial market trends is a close collaborative process between a wide variety of teams within Amundi. The underlying proprietary platform generating these asset prices is named Cascade Asset Simulation Model (CASM), and was originally developed in a joint venture between leading practitioners from Amundi (previously Pioneer Investments) and Cambridge University. The platform mirrors industry and academic best practices, taking into account the inherent complex relationships between macroeconomics, monetary policy and market dynamics. The CASM platform covers macro and financial variables for major regions, in particular the US, UK, Eurozone, Japan, China and Emerging Markets as an aggregate.

Applications of Monte Carlo methods in asset price simulation are well documented by Glasserman (2003), and Hull (2021) among others. Stochastic generation of risk factor scenarios permits us to consider a wide range of possible asset prices and control the uncertainty surrounding these. The additional flexibility allows us to vary starting assumptions and see the effect asset price scenarios. This possibility enhances our ability to simulate coherent scenarios across any instrument in a multi-asset portfolio and an asset-liability framework, a feature that is particularly relevant to institutional clients with long time horizons – see examples from Dempster *et al.* (2007, 2009).

The architecture of CASM can be broadly defined in two dimensions.



Source: Amundi Asset Management CASM Model - Institutional Advisory, for illustrative purposes only.

- The first dimension is composed of a top-down "*cascading*" effect (hence the name Cascade Asset Simulation Model), starting at the top with macroeconomic variables, reflected subsequently in the financial market risk factors. Initially proposed by Wilkie (1984) and further developed by Dempster *et al.* (2009), this cascade structure is at the root of the platform's interdependent linear and non-linear relationships between risk factors, which are ultimately used to define the prices of the different assets and financial instruments.

- The second dimension of CASM portrays a representation of the future evolution of the aforementioned "*cascading*" effect. The unique formulation allows for simulation of coherent scenarios between the different risk factors from the short to the long time horizons. In the short term, CASM blends econometric models and quantitative short-term outlooks from in-house practitioners.

Periodic and on-demand reviews along the two dimensions by in-house specialists enable us to make necessary adjustments without significant re-formulation or re-engineering of the platform, resulting in both consistent mean scenarios and efficient parameter control in terms of estimates and transparency of explanatory variables. The current implementation follows the best practice in implementing complex and both linear and non-linear relationships between any number of market and economic factors available. Empirical studies and qualitative views do in fact have a significant effect on their medium-term behaviour as they converge to the predetermined equilibrium levels that are inferred by our models and enhanced by qualitative judgement of Amundi's experts. The model's main aim is not to forecast short-term trends with precision, but to give a good representation of the medium and long-term evolution of the financial variables considering pairwise correlations and riskiness.

The following dependency matrix shows the dependencies between models where $\rightarrow$ in a cell indicates that the row model depends on the column model.

	GDP Cycle	Cmdty Cycle	Real Rate	EPS Cycle	Inflation	Short Rate	Slope	Long Rate	Liquidity	GARCH	Equity	Rental Yield
Inflation	$\rightarrow$	$\rightarrow$										
Short Rate	$\rightarrow$		$\rightarrow$		$\rightarrow$							
Slope						$\rightarrow$						
Long Rate						$\rightarrow$	$\rightarrow$					
Term Structure						$\rightarrow$	$\rightarrow$	$\rightarrow$				
Credit Spread	$\rightarrow$								$\rightarrow$			
Country Spread	$\rightarrow$								$\rightarrow$			
Equity	$\rightarrow$			$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$		$\rightarrow$	$\rightarrow$		
Private Equity									$\rightarrow$		$\rightarrow$	
Rental Yield	$\rightarrow$				$\rightarrow$							
Real Estate									$\rightarrow$		$\rightarrow$	$\rightarrow$
Infrastructure					$\rightarrow$				$\rightarrow$			

Source: Amundi Asset Management

For each model listed above, a series of innovations are generated. For subsequent models, a correlation matrix of innovations for the relevant variables (e.g. GDP cycle and commodity cycle for the inflation model) is used to generate a series of correlated random numbers via the Cholesky decomposition. In this manner, the model captures the causal relationship of the cascade structure and the interdependence between the risk factors.

Simulation & Model Structure Under a Monte Carlo environment, scenarios are generated along the aforementioned two dimensions taking into consideration the historical and simulated correlation among the respective risk factors. In this manner, we are able to model the distribution of the variables at each time step by matching the simulated and historical statistical moments. A key factor to an effective and useful simulation platform is its ability to portray a meaningful representation of the future evolution of financial risk factors. Our belief is that risk factors will evolve towards their respective dynamic trend over a medium and long-term time horizon. Under this framework, the short-term outlook of each risk factor is an exogenous input which limits the set of admissible parameters in the calibration process in order to determine the initial direction and the reversion speed towards trend. The resulting risk factor evolution is designed to match the corresponding official short-term outlook. On the long-term time horizon, we use best-practice statistical and economic properties to derive an intuitive equilibrium level for each of the risk factors in question. The resulting equilibrium levels enhance the stability and statistical interpretation of the simulations. Meanwhile, the medium-term dynamics are primarily driven by variables related to business cycles and serve to link the short-term and long-term risk factor levels.

The prevailing quantitative formulation governing CASM is that of a multi-factor diffusive mean reverting process. As previously mentioned, models in the platform exhibit long-run behaviour converging to predetermined long run levels or dynamic trends. The convergence process is governed both by the distance between the current endogenous variable level and the equilibrium level and the future evolution of the business cycle related variables. Academic and empirical studies point to the existence of such mean reversion behaviour in the markets – see Uhlenbeck and Ornstein (1930) and Chan *et al.* (1992) for an exposition on this topic. For each of the distinct risk factor models in CASM, the calibration process estimates parameters by minimizing the distance (in a least squares sense) between the statistics that describe the distribution of the historical data and the statistics of the sampling distribution from the simulated data. The cascade model structure captures the complex relationships between macro-economic and financial variables. The resulting simulated multivariate distributions exhibit tail co-dependence and skewness that we observe empirically. Further alignment of the models with their historical correlation estimates is achieved by correlating the random numbers of the stochastic processes.

Important Information

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