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The Management of Retirement Savings revisited

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His main fields are portfolio optimization, asset management, wealth management and tax incidence (in particular on investment return).

Abstract

Retirement is not the only motivation of saving but it is a prominent one. Whether channelled through pension funds or individual accounts, the question of how to allocate retirement savings, and in particular which degree of risk to tolerate, is fundamental.

Investing in risky assets should not be viewed as a way to compensate for insufficient savings during a life time, or a way to optimise the likelihood of reaching a future consumption target, whatever the consequences in bad circumstances.

However, as risk free assets tend to vanish, or yield negative returns, investing in risky assets is a way to improve expected returns on savings, and thus expected purchasing power at old age, provided the cost of risk may be mitigated. One way of increasing the tolerance to investment risk is the potential stream of future labour income if there is some flexibility on the retirement departure age or the possibility having a job (full or part-time) during the first years of retirement.

With reasonable parameters, such flexibility provides a significant incentive to increase investment in risky assets and provide significant welfare gains. Finally, labour supply flexibility gives a reason for the optimal share of risky assets to decline with age.

Keyword: Savings, retirement, asset allocation, portfolio management, risk

JEL Classification: E21, G11, G23

1. Introduction

The return on risk-free assets, if such assets truly exist, is very low today and will undoubtedly remain so for a long period of time. Therefore, if investment is made only in risk-free assets or moderately risky assets, savings for retirement will procure low purchasing power when it comes time to retire and will be insufficient for reaching any consumption targets (such as maintaining a certain lifestyle).

If wealth is invested in risky assets, the expectations of return are higher; however, there is a risk factor: in some adverse configurations, purchasing power could be perceived as catastrophically low.

Risky investing is not an appropriate way to make up for insufficient savings[1]. When risk is rewarded, risk-taking can increase the likelihood of a reaching a minimum purchasing power target in retirement but at the price of an exposure to adverse circumstances, which risk measures such as VaR or CVaR accurately capture.

In the specific case of saving for retirement, the individuals concerned have, however, a way to mitigate the consequences of an unfavourable configuration for return on investment in risky assets: supplementing inadequate purchasing power with labour income. In practice, such income can be secured by putting off retirement or by getting a job, even if part-time, during the first few years of retirement.

In this research paper, we will simultaneously model, with conventional utility and expected utility functions, the labour supply at the time of retirement and the portfolio allocation choice for retirement savings. We examine the importance of labour supply flexibility and the impact of such flexibility on asset allocation. We find that that the portion allocated to risky assets can be substantially increased.

This result is consistent with the work done on labour supply flexibility and portfolio choice (Bodie, Merton and Samuelson, 1992) and generalizes it for different levels of risk tolerance (or aversion).

In light of these results, we can at last give thought to the identification of pension fund commitments, or objectives.

[1] As Zvi Bodie has shown, expressions such as *I cannot afford not to invest in risky assets* should be banished.

2. Characteristics of building savings for retirement

Retirement can be defined as a period in life in which an individual no longer receives income from his or her profession or labour. This individual's consumption needs will therefore have to be covered by other funds: either transfers or labour income earned prior to retirement that was not consumed immediately and was therefore saved. Saving is therefore a key factor in having funds available and consuming during retirement.

Conversely, retirement is a motivation to amass savings during a person's working life. It is not the sole motivation – people also save as a precaution, to meet temporary interruptions in labour income, and to satisfy the desire to transmit purchasing power to their heirs – but it is an important motivation.

One characteristic of saving for retirement is the length of the time horizon (delayed purchasing power). Usually retirement lasts 20 years and follows forty years of work. On average, a period of approximately thirty years elapses between the time savings are amassed and when they are used for consumption.

This length varies depending on the age of the working individual of interest: for a young person entering the workforce, this period is almost fifty years – a half-century. For an individual about to enter retirement, it would be ten years or so.

There are two principal schemes for saving for retirement: an individual format and an institutional format, which we will refer to generically as pension funds. Under the individual format, the saver (assisted by his or her advisers) has primary responsibility for asset allocation. Under the institutional format, the fund itself is responsible for asset allocation. This does not prevent that at the end of the day it is usually the saver who is impacted by the consequences of the choices made, with one major exception: defined benefit retirement plans guaranteed by a sponsor who is often the individual's employer. In this case, risk is ultimately borne by the sponsor (except in the event of bankruptcy) and must be managed within the set of risks to which he is exposed.

All methods combined, savings for retirement must be substantial. The targeted goal is often defined as a ratio –50% to 70%– of the benefit received to the labour income in the last years of employment or sometimes the benefit received to the average of labour income over the working life.

In fact, it is achievable consumption that is the aim, so that a certain lifestyle can be maintained after retirement, at least partially. Arguments are made that the needs of consumption are lower after retirement due to, among other reasons, children leaving the household. But there are also arguments in favour of aiming at higher level of resources to cover care and medical bills.

With an actual rate of return (after tax[2]) on investment near zero, which is optimistic today for risk-free savings, approximately one-third of all labour income must be set aside for savings, with a desirable replacement rate of two-thirds.

Table 1

The savings effort required based on real return

T = 30 years, replacement rate = 2/3

Real return	Required Saving Rate (a)	Required Saving Rate (b)
-2%	0.611	0.379
0%	0.333	0.250
2%	0.184	0.155
4%	0.103	0.093
6%	0.058	0.055

(a) : Replacement rate as a proportion of working life income

(b) : Replacement rate as a proportion of working life income net of retirement savings

This assessment is made using a two-period model, assuming that savings is amassed mid-career and its fruits spent midway through the retirement period. In practice, the model should be fine-tuned to take account of the characteristics of the labour income time profile and mortality tables. It is worth bearing in mind that the orders of magnitude obtained are nonetheless significant.

Savings here should be understood in a sense widened to include contributions to mandatory plans, in particular through pay-as-you-go regimes. Naturally, the savings effort is very dependent on expected real return. This is obviously very important for risk-free investing, where expected real return is very low and sometimes even negative. The effort can be substantially reduced for higher real returns but they can only be achieved through risk-taking.

[2] Pension funds and pension accounts usually work under tax neutrality: contributions are deductible from the income tax base and pensions when retrieved are added to it. If income tax rates do not differ between the two periods, tax does not impact real return. With other saving channels, tax generally eats into real returns, the more so inflation is high (Maillard, 2011b).

In case of low real returns, the saving effort is so huge that it is certainly fair to compute the ratio on a net-of-saving basis, to express the target in terms of old-age potential consumption as a fraction of working age potential consumption, or working age income net of retirement savings (column (b) in Table 1). That reduces somewhat the savings ratio target but it remains important for low returns.

3. The question of managing savings for retirement

3.1. Limitations on management in terms of target.

3.1.1. The issue

In this section, we assume that savings have been accumulated and that the individual has, at moment in time 0, wealth or capital, W_0, that he or she must invest in one way or another (or that must be invested on their behalf).

The time horizon to retirement is T. We assume that are efficient mechanisms to transform wealth, W_T, secured on this horizon, into annuities[3], and that this final capital reflects the degree of achievement of this person's goals.

If there is a risk-free investment between dates *0* and T, and if the initial wealth is invested in this risk-free asset, the capital secured is known with certainty. However, most often, the initial wealth will be invested, as least in part, in risky assets and the capital secured will be exposed to risk.

Pension funds have often obligations (in the case of defined benefit plans) or explicit or implicit objectives of paying specified amounts, defined nominally or in purchasing power, i.e. W_T*. If savings is not managed by an institution, the individual may also think in terms of a target.

Commitments will be kept, or the targets reached if:

$$W_T \geq W_T\,*$$

There are two possible cases. Risk-free investment (nominal or corrected for inflation) can produce a return, r_f, sufficient to comply with commitments or reach targets.

[3] The hypothesis is therefore formulated that pensions are not exposed to risk once liquidated, which undoubtedly satisfies the desires of most retirees.

$$W_0 e^{r_f T} \geq W_T *$$

The question of asset allocation is straightforward in that case. It can consist of investing a portion of wealth, $e^{-r_f T} W_T *$, in risk-free assets, and the remainder in high-risk assets. The proceeds from investment in this portion could be disbursed as a pension bonus or used to reimburse contributions by the retirement plan's sponsor.

The first example is perhaps not the most typical, especially in the early 2010s. Often it is impossible to meet commitments or to reach targets by means of a risk-free asset.

$$W_0 e^{r_f T} < W_T *$$

Complying with commitments or reaching targets is no longer certain. How then should be question of asset allocation be formulated?

Should the probability of complying with commitments be maximised, or, by the same token, should the chances of default of the fund due to its obligations be minimised?

$$Max((\Pr(W_T > W_T *))$$

Formalisation of this type could create more room for risky assets (the probability of complying with obligations is zero if investments are made solely in risk-free assets). But if we go deeper, it could lead to renouncing all *upside* in excess of the W_T* limit and hence to selling *puts* at this threshold on the portfolio of risky assets. Going beyond that, the optimum solution would be found in investing in a binary option backed by a portfolio of risky assets, paying 0 with a low probability, and exactly W_T* with the highest possible probability.

The drawback of formalising by minimising the probability of default is clear at this stage: missing targets is not punished in a manner specific to the degree of failure reflecting the shortfall between the target and actual performance.

Progressive penalisation based on such a shortfall should be introduced and in a manner consistent with the cost inflicted on the pension beneficiaries and which would take account of the fact that these beneficiaries can mitigate this cost by virtue of increased labour supply. We come back to the problem of optimising for the benefit of the ultimate investor as the pension fund is transparent (but ensures, of course, the functions of allocating the lifetime risk and handling the financial administration of the savings).

We therefore assume that the charge of the shortfall does not land on the shoulders of the plan sponsor. We also do not take into account fund's possible regulatory constraints on asset allocation.

3.1.2. Modelling

Wealth can be invested in risk-free assets, if one assumes they exist, or it can be invested in an optimally-managed risk portfolio[4] (instantly optimising the Sharpe ratio) if adopting a dynamic management style[5]. Here μ is average annual expectation of return over the risk portfolio period, σ is annualised volatility and t is the Sharpe ratio.

$$t = \frac{\mu - r_f}{\sigma}$$

The eventual value of the wealth, if a portion, α, is (continually) invested in the risk portfolio, is:

$$W_T = W_0 e^{\left[r_f + \alpha(\mu - r_f) - \frac{1}{2}\alpha^2\sigma^2\right]T + \alpha\sigma\sqrt{T}\varepsilon}$$

where ε is a random variable with zero mean and unitary standard deviation. In the interests of simplicity, we will use a Gaussian risk distribution, in particular because of the distant horizon.

The target will be reached or exceeded if:

$$\varepsilon \geq \varepsilon^* = \left[\frac{r^* - r_f}{\alpha\sigma} - t - \frac{1}{2}\alpha\sigma\right]\sqrt{T}$$

$$\Pr(W_T > W^*_T) = 1 - \Phi(\varepsilon^*)$$

Φ is the law of cumulative frequency distribution of the risk.

r^* is the risk-free rate of return that must be secured in order to reach the target with a 100% investment in risk-free assets.

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[4] The portfolio of risky assets is not necessarily the market portfolio. Room is left for active portfolio management to improve the performance of the risk portfolio.

[5] We are speaking of "lite" dynamic allocation in the sense that the parameters are deemed constant over the period (no predictability). Its results will differ very little from that of static allocation, with sufficient risk aversion (Maillard, 2011). The choice of dynamic allocation allows an analytic treatment of the optimization problem.

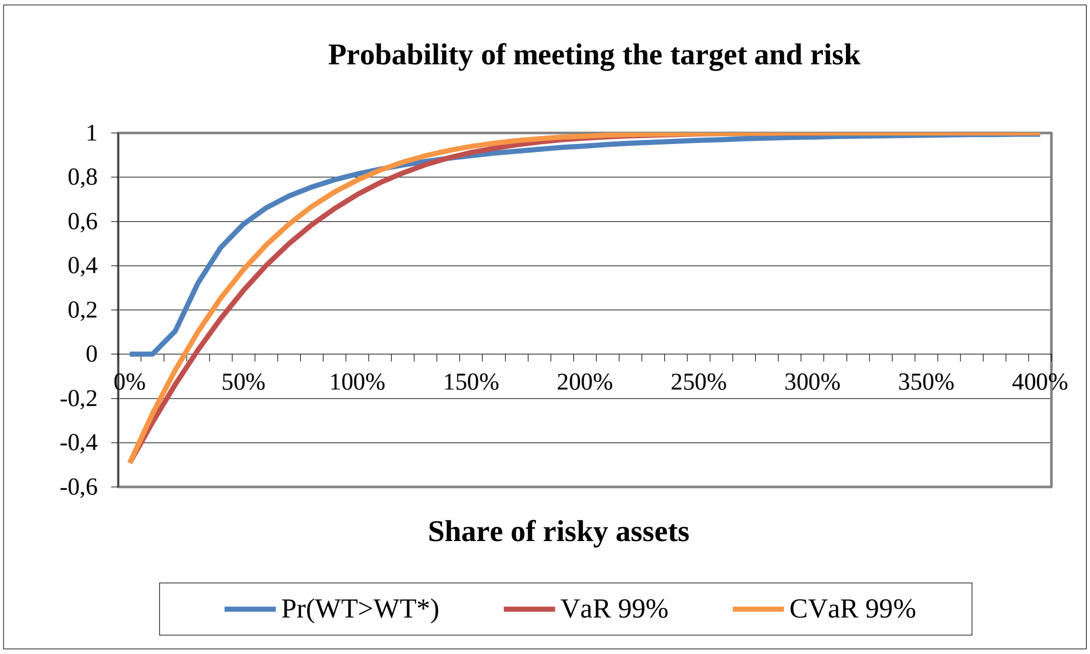
If this threshold performance is less than the actual risk free rate, investing all wealth in a risk-free asset is sufficient to achieve a probability of 1 in reaching the target (trivial, $\alpha = 0$, $\varepsilon^* = -\infty$)

If it is greater, there is a *funding gap* at the effective risk free rate and the difference represents a shortfall in annualized returns. The threshold therefore decreases with the portion of risky assets. The theoretical optimum is found, for an infinite investment in the risk portfolio, with leverage over the risk-free asset.

The quid pro quo is obviously risk, which can be assessed using conventional measurements, the standard deviation of final wealth or Value-at-Risk (VaR) or Conditional Value-at-Risk (CvaR, aka Expected Shortfall).

As an illustration, the chart below provides a representation of the parallel change in the probability of reaching a target and VaR and CVaR at a threshold of 99% in proportion to the investment, with an investment term of 20 years, a risk free rate of 2%, a credit spread of 4%, a annualised volatility of the risk portfolio of 20% and a *funding gap* of 2% per year.

Chart 1



Given the limitations of reasoning in terms of targets, it is logical to place the asset allocation question within a framework of optimisation.

3.2. Management in response to optimisation

3.2.1. No flexibility in labour supply after retirement

We use the classic Neumann-Morgenstern framework to optimise the expected utility delivered by consumption derived from purchasing power of the value of the accumulated savings at the time of retirement.

$$MaxE(U(W_T))$$

As to the form of the utility function, we have opted for a function with the feature of constant relative risk aversion (CRRA). In the catalogue of standard utility functions, we have rejected the quadratic function (with which utility decreases with consumption after a certain threshold), and the constant absolute risk aversion (CARA) function. With this last class of functions, the optimal proportion of risky assets decreases with the initial wealth of the savers, which is counter-intuitive (See Annex 4). We do not retain a hyperbolic absolute risk aversion (HARA) function due to the difficulty in identifying irreducible consumption and distinguishing if from the "targets" described above. However, we will use the results related to optimisation and describe how to adjust our results if a hyperbolic risk aversion function were used (Annex 4).

The utility function eventually retained has the form:

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma}$$

The range usually deemed realistic for risk aversion is along the lines of 2 to 7/10.

The expected utility is maximised (see Annex 2) for:

$$\alpha = \frac{\mu - r_f}{\gamma \sigma^2}$$

Using this value for the portion of risky assets:

$$E(U(W_T)) = (1-\gamma)^{-1} {W_0}^{1-\gamma} e^{(1-\gamma)(r_f + s)T}$$
$$with \quad s = \frac{t^2}{2\gamma}$$

Compared to a risk-free investment, the optimisation of asset allocation leads to a better utility expectation equivalent to risk-free supplemental return equal to the square of the Sharpe ratio divided by double the aversion coefficient related to risk.

3.2.2. Flexibility in the labour supply on retirement

To take into account labour supply flexibility, the utility function is augmented to:

$$U(C,L)=\frac{C^{1-\gamma}}{1-\gamma}+b\frac{(\underline{L}-L)^{1-\gamma}}{1-\gamma}$$

C is the consumption for the period, L is the labour supplied, $\underline{L}$ is the maximum amount of work that is possible to provide and $\underline{L}$ - L is therefore leisure.

The first term is the conventional utility function (CRRA) with constant relative aversion equal to γ. The second term represents the contribution of leisure to total utility, with a weighting dependent on coefficient *b*.

Using the same exponent for consumption and for leisure firstly cross-references a conventional utility function of the constant elasticity of substitution (CES) type. The elasticity of substitution between consumption and leisure is:

$$s=\frac{1}{1-(1-\gamma)}=\frac{1}{\gamma}$$

As relative aversion to risk is generally greater than 1, the elasticity of substitution is confined in a range 0 to 1, which is reasonable.

Furthermore, using the same exponent provides an analytical solution to the problem of optimising utility, at the moment of interest, under a budget constraint.

The funds available for spending are in fact made up of accumulated wealth, *W*, and labour income. If *w* represents the labour compensation rate, optimization is written:

$$Max U(W+wL,L)$$

And leads to labour supply at retirement age equal to:

$$L=\frac{b^{-\frac{1}{\gamma}}}{b^{-\frac{1}{\gamma}}+w^{1-\frac{1}{\gamma}}}\underline{L}-\frac{w^{-\frac{1}{\gamma}}}{b^{-\frac{1}{\gamma}}+w^{1-\frac{1}{\gamma}}}W$$

The supply of labour decreases linearly with accumulated wealth[6].

Entering this value for labour supply into the utility function gives (see Annex 3):

[6] The labor supply derived from this formula can be negative, which is tolerable: if performance of the risky assets is very strong, the individual can move forward (if rules permit) the age at which he or she can receive his or her pension.

$$U(C,L)=\frac{(W+w\underline{L})^{1-\gamma}}{1-\gamma}\left(1+b^{\frac{1}{\gamma}}w^{1-\frac{1}{\gamma}}\right)^{\gamma}=V(W)$$

The function *V(W)* is a HARA-type function, apart from the fact that the term is constant, representing minimum consumption, is normally negative in this type of function. Here the constant term is positive and represents the maximum value of supplemental labour income that the retiree can secure by sacrificing all his or her leisure.

The second phase entails maximising *V(W)*. We can use the results of Bajeux, Jordan and Portait (2003) when maximising HARA utility. In this case, the optimal allocation is the following combination:

- a risk-free investment ultimately yielding exactly the minimum consumption,
- a dynamic portfolio optimally combining risky assets and risk-free assets, the same as that resulting from optimising a CRRA.

For the question at issue, optimal allocation is composed of:

- a short position in the risk free asset corresponding to the present value, at a risk free rate, of the maximum labour income that the retiree can receive, or $-w\underline{L}e^{-r_f T}$
- a long position in the portfolio combining the risk portfolio and the risk-free asset using dynamic management.

The proportion of the risky assets will change over time with the value of the portfolio. At the outset, it is:

$$\alpha\frac{W_0+w\underline{L}e^{-r_f T}}{W_0}=\alpha\left(1+\frac{w\underline{L}e^{-r_f T}}{W_0}\right)$$

The portion invested in risky assets is increased by the ratio of the present value of potential earnings from future work to savings accumulated for retirement. As accumulated savings usually grows with the age of the saver, while the earnings from potential future work are independent, the optimal portion of the risky assets therefore decreases with age.

The impact of flexibility on the risk asset portion is not immaterial. To demonstrate this point, let's take an individual mid-way through his or her working life and for whom accumulated savings represents four times the labour income (20 years times 20% the savings rate), assumed to be constant. If this person can envisage working the equivalent of two years subject to the same pay conditions and assuming the risk free rate is zero, the optimal proportion of risky assets is increased by half.

It is finally possible to compute the welfare gains of labour supply flexibility (see Annex 3) by comparing the expected utility with flexibility to the expected utility without ($L = 0$). Part of those gains stem from the ability to invest more in risky assets that labour supply flexibility provides.

4. Discussion and conclusions

We have shown, using a simple model without being unrealistic, that labour supply flexibility at the time of retirement can improve economic welfare, both directly and indirectly, by making it possible to invest in riskier assets and to capture the rewards of risk.

In fact, it happens that working during retirement or deferring retirement procures additional wealth at no risk or, in any event, with little risk (but not without pain…). This assumes that there are good assurances against potential joblessness and, above all, against the inability to work. The optimisation question ultimately translates into optimisation with the constraint of a fixed investment in a given asset.

As to pension funds, and especially in the case where they manage most of the retirement savings of their principals, we would recommend promoting flexibility on the liquidation age. The ideal would be to offer à la carte allocation, taking account of risk aversion on the one hand and, on the other, the opportunity and willingness to get an old-age job or to defer retirement.

References

Amenc, Noël, Felix Goltz, Lionel Martellini and Vincent Milhau, 2010, “New Frontiers in Benchmarking and Asset Liability Management”, Working Paper, Edhec Risk Institute, September 2010

Bajeux-Besnainou, Isabelle, James V. Jordan and Roland Portait, 2003, "Dynamic Asset Allocation for Bonds, Stocks and Cash", Journal of Business, Apr 2003, Vol. 76, Issue 2

Bodie, Z., R. C. Merton and W. F. Samuelson, 1992, “Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model,” *Journal of Economic Dynamics and Control*, 16, 427-449

Bodie, Z., 2001, "Retirement Investing": A New Approach, Working Paper 2001-03, Boston University School of Management

Maillard, Didier, 2011a, “Dynamic versus Static Asset Allocation: From Theory (halfway) to Practice”, Working Papers Series N° 1942901, SSRN, March 2011

Maillard, Didier, 2011b, “Tax and Investment Return”, Working Papers Series N° 1968985, SSRN, September 2011

Annex 1

Target is achieved iff:

$$W_T = W_0 e^{\left[r_f + \alpha(\mu - r_f) - \frac{1}{2}\alpha^2\sigma^2\right]T + \alpha\sigma\sqrt{T}\varepsilon} \geq W_T *$$

$$\left[r_f + \alpha(\mu - r_f) - \frac{1}{2}\alpha^2\sigma^2\right]T + \alpha\sigma\sqrt{T}\varepsilon \geq \ln\frac{W_T *}{W_0}$$

$$\varepsilon \geq \frac{\ln\frac{W_T *}{W_0} - \left[r_f + \alpha(\mu - r_f) - \frac{1}{2}\alpha^2\sigma^2\right]T}{\alpha\sigma\sqrt{T}} = \left[\frac{r * - r_f}{\alpha\sigma} - t - \frac{1}{2}\alpha\sigma\right]\sqrt{T} = \varepsilon *$$

$$\Pr(W_T > W *_T) = 1 - \Phi(\varepsilon *)$$

Annex 2

If a portion of wealth α is invested in a risk portfolio with an expected return of μ and volatility on this return of σ, and a portion, 1-α, is invested in a risk-free asset whose return is r, the relative variation in wealth over time can be represented by the following equation:

$$\frac{dW}{W} = (1-\alpha)r_f\, dt + \alpha(\mu dt + \sigma dz) = \left[r_f + \alpha(\mu - r_f)\right]dt + \alpha\sigma dz$$

where dz is standard Brownian motion.

Using Itō's lemma and integrating leads to:

$$d\ln W = \left[r + \alpha(\mu - r_f) - \frac{1}{2}\alpha^2\sigma^2\right]dt + \alpha\sigma dz$$

Assuming the random process is Gaussian, and ε designing a zero mean unitary variance random variable

$$W(T) = W(0)e^{\left[r_f + \alpha(\mu - r_f) - \frac{1}{2}\alpha^2\sigma^2\right]T + \alpha\sigma\sqrt{T}\varepsilon} = W_T = W_0 e^{\left[r_f + \alpha(\mu - r_f) - \frac{1}{2}\alpha^2\sigma^2\right]T + \alpha\sigma\sqrt{T}\varepsilon}$$

If we assume that the utility function is CRRA with parameter γ,

$$U(W) = (1-\gamma)^{-1}W^{1-\gamma}$$

$$U(W_T) = (1-\gamma)^{-1}W_0^{1-\gamma}e^{(1-\gamma)(r_f + \alpha(\mu - r) - \alpha^2\sigma^2/2)T}e^{(1-\gamma)\alpha\sigma\sqrt{T}\varepsilon}$$

$$E(U(W_T)) = (1-\gamma)^{-1}W_0^{1-\gamma}e^{(1-\gamma)(r_f + \alpha(\mu - r) - \alpha^2\sigma^2/2)T}e^{(1-\gamma)^2\alpha^2\sigma^2 T/2}$$

$$E(U(W_T)) = (1-\gamma)^{-1}W_0^{1-\gamma}e^{(1-\gamma)(r_f + \alpha(\mu - r_f))T - (1-\gamma)\alpha^2\sigma^2/2)T + (1-\gamma)^2\alpha^2\sigma^2/2)T}$$

$$E(U(W_T)) = (1-\gamma)^{-1}W_0^{1-\gamma}e^{(1-\gamma)(r_f + \alpha(\mu - r_f) - \gamma\alpha^2\sigma^2/2)T}$$

In fact, expected utility is the same as that obtained in the case of a risk-free investment, for which the return would be:

$$r_c = r_f + \alpha(\mu - r_f) - \gamma\alpha^2\sigma^2/2 = E(r_P) - \frac{\gamma}{2}V(r_P)$$

Expected utility is maximised for:

$$\alpha = \frac{\mu - r_f}{\gamma\sigma^2}$$

Using this value for the portion of risky assets:

$$r_f + \alpha(\mu - r_f) - \gamma\alpha^2\sigma^2/2 = r + \frac{(\mu - r_f)^2}{\gamma\sigma^2} - \gamma\frac{(\mu - r_f)^2}{\gamma^2\sigma^4}\frac{\sigma^2}{\sigma^2} = r_f + \frac{1}{2}\frac{(\mu - r_f)^2}{\gamma\sigma^2} = r_f + \frac{1}{2}\frac{t^2}{\gamma}$$

$$E(U(W_T)) = (1-\gamma)^{-1}W_0^{1-\gamma}e^{(1-\gamma)(r_f + t^2/2\gamma)T} = (1-\gamma)^{-1}\overline{W_0}^{1-\gamma}e^{(1-\gamma)r_f T}$$

$$e^{sT} = \frac{W_0}{\overline{\overline{W_0}}} = \left[e^{(1-\gamma)(t^2/2\gamma)T}\right]^{\frac{1}{1-\gamma}} = e^{(t^2/2\gamma)}$$

$$s = \frac{t^2}{2\gamma}$$

Annex 3

1) Optimization of labour supply

$$MaxU(W + wL, L) = \frac{(W + wL)^{1-\gamma}}{1-\gamma} + b\frac{(\underline{L} - L)^{1-\gamma}}{1-\gamma}$$

$$\frac{dU}{dL} = w(W + wL)^{-\gamma} - b(\underline{L} - L)^{-\gamma} = 0$$

$$w^{-\frac{1}{\gamma}}(W + wL) = b^{-\frac{1}{\gamma}}(\underline{L} - L)$$

$$L\left[b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}\right] = b^{-\frac{1}{\gamma}}\underline{L} - w^{-\frac{1}{\gamma}}W$$

$$L = \frac{b^{-\frac{1}{\gamma}}}{b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}}\underline{L} - \frac{w^{-\frac{1}{\gamma}}}{b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}}W$$

2) Resources at old-age

$$W + wL = \frac{b^{-\frac{1}{\gamma}}}{b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}} w\underline{L} + \left(1 - w\frac{w^{-\frac{1}{\gamma}}}{b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}}\right) W = \frac{b^{-\frac{1}{\gamma}}}{b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}} w\underline{L} + \frac{b^{-\frac{1}{\gamma}}}{b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}} W$$

$$W + wL = \frac{b^{-\frac{1}{\gamma}}}{b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}} (W + w\underline{L})$$

$$\underline{L} - L = \left(1 - \frac{b^{-\frac{1}{\gamma}}}{b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}}\right)\underline{L} + \frac{w^{-\frac{1}{\gamma}}}{b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}} W = \frac{w^{-\frac{1}{\gamma}}}{b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}} (W + w\underline{L})$$

3) Utility function

$$U(C,L) = \frac{1}{1-\gamma}\left[\left(\frac{b^{-\frac{1}{\gamma}}}{b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}}\right)^{1-\gamma} (W + w\underline{L})^{1-\gamma} + b\left(\frac{w^{-\frac{1}{\gamma}}}{b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}}\right)^{1-\gamma} (W + w\underline{L})^{1-\gamma}\right]$$

$$U(C,L) = \frac{1}{1-\gamma}\left[b^{-\frac{1-\gamma}{\gamma}}\left(\frac{1}{b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}}\right)^{1-\gamma} + bw^{-\frac{1-\gamma}{\gamma}}\left(\frac{1}{b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}}\right)^{1-\gamma}\right](W + w\underline{L})^{1-\gamma}$$

$$U(C,L) = \frac{(W + w\underline{L})^{1-\gamma}}{1-\gamma} b\left(b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}\right)\left(\frac{1}{b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}}\right)^{1-\gamma} = \frac{(W + w\underline{L})^{1-\gamma}}{1-\gamma} b\left(b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}\right)^{\gamma}$$

$$U(C,L) = \frac{(W + w\underline{L})^{1-\gamma}}{1-\gamma}\left(1 + b^{\frac{1}{\gamma}} w^{1-\frac{1}{\gamma}}\right)^{\gamma} = V(W)$$

Optimally, final wealth is:

$$W_T = \left(W_0 + w\underline{L}e^{-r_f T}\right)e^{\left[r_f + \alpha(\mu - r_f) - \frac{1}{2}\alpha^2\sigma^2\right]T + \alpha\sigma\sqrt{T}\varepsilon} - w\underline{L}$$

As to the funds available for consumption, they are

$$W_T + wL = \frac{b^{-\frac{1}{\gamma}}}{b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}}(W_T + w\underline{L}) = \frac{b^{-\frac{1}{\gamma}}}{b^{-\frac{1}{\gamma}} + w^{1-\frac{1}{\gamma}}}\left(W_0 + w\underline{L}e^{-r_f T}\right)e^{\left[r_f + \alpha(\mu - r_f) - \frac{1}{2}\alpha^2\sigma^2\right]T + \alpha\sigma\sqrt{T}\varepsilon}$$

The utility is:

$$U(C,L)=\frac{(W_T+w\underline{L})^{1-\gamma}}{1-\gamma}\left(1+b^{\frac{1}{\gamma}}w^{1-\frac{1}{\gamma}}\right)^{\gamma}$$

$$=(1-\gamma)^{-1}\left(1+b^{\frac{1}{\gamma}}w^{1-\frac{1}{\gamma}}\right)^{\gamma}\left(W_0+w\underline{L}e^{-r_fT}\right)^{1-\gamma}e^{(1-\gamma)\left(\left[r_f+\alpha(\mu-r_f)-\frac{1}{2}\alpha^2\sigma^2\right]T+\alpha\sigma\sqrt{T}\varepsilon\right)}$$

$$E(U)=(1-\gamma)^{-1}\left(1+b^{\frac{1}{\gamma}}w^{1-\frac{1}{\gamma}}\right)^{\gamma}\left(W_0+w\underline{L}e^{-r_fT}\right)^{1-\gamma}e^{(1-\gamma)(r_f+t^2/2\gamma)T}$$

If there is no labour flexibility (L = 0), the expected utility is

$$E(U)=(1-\gamma)^{-1}\left[(W_0)^{1-\gamma}e^{(1-\gamma)(r_f+t^2/2\gamma)T}+b\underline{L}^{1-\gamma}\right]$$

We can measure the gains in welfare associated with flexibility by comparing the savings leading to expected utility with flexibility and the higher savings needed to reach the same degree of utility without flexibility.

Expressed in monetary terms, the resulting gain in welfare due to flexibility is:

$$\overline{W}_0-W_0$$

$$(1-\gamma)^{-1}\left[\overline{W}_0{}^{1-\gamma}e^{(1-\gamma)(r_f+t^2/2\gamma)T}+b\underline{L}^{1-\gamma}\right]=(1-\gamma)^{-1}\left(1+b^{\frac{1}{\gamma}}w^{1-\frac{1}{\gamma}}\right)^{\gamma}\left(W_0+w\underline{L}e^{-r_fT}\right)^{1-\gamma}e^{(1-\gamma)(r_f+t^2/2\gamma)T}$$

$$u=r_f+\frac{t^2}{2}\quad A(w)=\left(1+b^{\frac{1}{\gamma}}w^{1-\frac{1}{\gamma}}\right)^{\gamma}$$

$$\left(\overline{W}_0e^{uT}\right)^{1-\gamma}+b\underline{L}^{1-\gamma}=A(w)\left((W_0+w\underline{L}e^{-r_fT})e^{uT}\right)^{1-\gamma}$$

$$\overline{W}_0=e^{-uT}\left[A(w)\left((W_0+w\underline{L}e^{-r_fT})e^{uT}\right)^{1-\gamma}-b\underline{L}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$$

Annex 4

Thoughts on the choice of the utility function

The utility functions commonly used are the quadratic function, the constant absolute risk aversion (CARA) function, the constant relative risk aversion function (CRRA) and the hyperbolic absolute risk aversion (HARA) function.

Quadratic function

It has the form:

$$U(C) = C - \frac{\varphi}{2}C^2$$
$$U'(C) = 1 - \varphi C$$

Consumption utility decreases for $C > 1/\varphi$, which is in conflict with the commonly accepted hypothesis that an abundance of wealth is not harmful.

CARA function

$$U(C) = e^{-aC}$$

Although it is used in the context of optimising a static portfolio (the result cannot be obtained analytically in the context of dynamic optimisation, but should not differ significantly), you obtain a result that is relatively counter-intuitive.

In fact, the final portfolio is worth, if α is the portion invested in the risk portfolio (with the notations in the body of the text):

$$W_T = W_0\left((1-\alpha)e^{r_f T} + \alpha e^{\mu T + \sigma\sqrt{T}\varepsilon}\right) = W_0\left(A + B\alpha\right) \quad A = e^{r_f T} \quad B = e^{\mu T + \sigma\sqrt{T}\varepsilon} - e^{r_f T}$$
$$U(W_T) = \exp[-a] = \exp[-aW_0\left(A + B\alpha\right)] = \exp[-aW_0 A]\exp[-aW_0 B\alpha]$$
$$E(U(W_T)) = \exp[-aW_0 A]E(\exp[-aW_0 B\alpha])$$

If $\alpha*$ maximises this expression, then $\alpha'* = \alpha * \dfrac{W_0}{W'_0}$ maximises

$$E(U(W'_T)) = \exp[-aW'_0\, A]E(\exp[-aW'_0\, B\alpha])$$

This means that the optimal percentage allocated to risky assets is inversely proportional to the initial wealth, which appears contrary to the perception of reality.

CARA function

It has the form:

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma}$$

HARA function

It has the form:

$$U(C) = \frac{(C-\hat{C})^{1-\gamma}}{1-\gamma}$$

$\hat{C}$ is irreducible consumption, below which the question of utility is irrelevant. It should not be confused with a target such as that discussed in the body of the article.

To maximise the expectation of a HARA function, on the one hand the risk-free asset must exist and, on the other, placing in this asset exactly what is needed to attain $\hat{C}$, the remainder being placed in a dynamic portfolio identical to that which would result from maximising a CRRA function with the same risk aversion parameter.

With labour flexibility, optimisation in the context of such a function would yield a solution whereby it would be necessary to place the difference (algebraic) between what is necessary to achieve $\hat{C}$ and the present value of the maximum of labour income in the risk-free asset.

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