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Tail Risk Adjusted Sharpe Ratio

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Tail Risk Adjusted Sharpe Ratio

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Abstract

The Sharpe Ratio has become a standard measure of portfolio management performance, taking into account the risk side. In that framework, the consideration of risk is reduced to returns volatility. The Sharpe Ratio does not encompass extreme risk, especially on the downside.

In this paper, we provide four methods for constructing Tail Risk Adjusted Sharpe Ratios (TRASR), using asymmetric measures of risk: semi variance, Value-at-Risk, Conditional Value-at-Risk (or Expected Shortfall) and expected utility derived measures of risks. We also have an exploration of the Aumann & Serrano's Economic Index of Riskiness.

The adjustment translates into an adjusted volatility, which allows us to keep the Sharpe Ratio format and to compare the impacts of the adjustments. For that, we use the Cornish-Fisher transformation to model tail risk, duly controlling for skewness and kurtosis.

Keywords: Risk, variance, volatility, skewness, kurtosis, portfolio performance

JEL Classification: C02, C51, G11, G32

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About the author

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Didier Maillard is Senior Advisor to Amundi on Research. He his Professor Emeritus at Conservatoire national des arts et metiers (CNAM), where he held a Chair of Banking. Previously, he has been an economist at the French Ministry of Finance and at the OECD (1980-1992) – economic forecasts, economic policy, public finance, tax studies, financial sector, and has occupied various positions at Paribas (and then BNP Paribas) from 1992 to 2001: chief economist, head of asset management, risk advisor. He is a graduate from Ecole polytechnique (Paris) and Ecole nationale d'administration.

His main fields are portfolio optimization, asset management, wealth management and tax incidence (in particular on investment return).

1 – Introduction

The Sharpe Ratio (SR) is a standard in the measurement of portfolio management performance, when risk is taken into account. The Sharpe ratio (Sharpe, 1966) is simply the ratio of an excess return (above a risk-free rate) to the volatility of returns. It may extended to the ratio of the absolute return to volatility if there is no such thing as a risk free asset. The risk-free asset may be replaced by a benchmark portfolio and the excess return compared to the benchmark divided by the volatility of the departures of returns compared to the benchmark, the tracking error. The Sharpe Ratio becomes an Information Ratio in such circumstances.

Volatility of returns is a crude measure of risk. It is symmetrical and does not put a special emphasis on very negative outcomes, which are assumed to be resented by investors. Given a mean return and a volatility, the Sharpe ratio is the same whatever the skewness and kurtosis of the distribution.

As a measure of performance, the Sharpe Ratio may be gamed by portfolio managers in various ways (Goetzmann & al., 2007), one of them being to build positions in negative skewness and excess kurtosis, the risk of which is not retraced in the Sharpe Ratio. Such risk premia do seem to exist, at least for negative skewness (Lempérière & al., 2014). In their study, the Sharpe ratio as traditionally defined seems to increase by ¼ times the negative skewness of the returns.

The association of negative skewness with a positive risk premium has also been explored by Feunou, Jahan-Parvar and Okou (2017), under the qualification of downside variance.

The shortcomings of the Sharpe ratio with respect to the description of risk are known for a long time. Some alternative measures taking into account the asymmetry of the returns distribution have been proposed: The Sortino ratio, Omega statistics and Kappa performance measures (Kaplan & Knowles, 2004). Goetzmann & al. propose a manipulation-proof performance measure (MPPM) which does take into account skewness and kurtosis. Their measure is very similar to that developed independently by Morningstar (2006).

The Sortino ratio is similar to the Sharpe ratio in that it divides a mean return by the square root of the sum of squared deviations from a threshold, but only when those are negative. If the threshold corresponds to the mean of distribution, that sum is often called the semi-variance.

Some authors (Amédée-Manesme & al., 2015, for example), divide the mean return by a Value-at-Risk (VaR) at a certain threshold.

The problem with those corrections is that one loses the scaling of the Sharpe ratio, a non-dimensional number the range of which academics and professionals are used to.

The aim of our paper is simple but goes a relatively long way: it is to provide a measure of risk-adjusted performance which coincides with the Sharpe ratio when the distribution of returns is Gaussian but which also takes into account tail risk when the distribution is not Gaussian. We will call it a tail-risk adjusted Sharpe ration, or TRaSR. It will be ratio of the mean performance to a tail-risk adjusted volatility.

We will then explore the impact of skewness and kurtosis on the measure by using the Cornish-Fisher transformation which allows to model distributions with a controlled tail risk.

2 – Adjusting the Sharpe ratio for tail risk

What we target is a measure that will coincide with the usual Sharpe Ratio when the returns are near to Gaussian; and that will impose a penalty when one is faced with extremely negative outcomes.

$$SR = \frac{\mu - r_f}{\sigma}$$

$$TRaSR = \frac{\mu - r_f}{\sigma_a}$$

The adjusted volatility will be equal to the actual volatility when the returns are Gaussian. It will be deigned to penalize negative skewness and excess kurtosis. We will build such adjusted volatilities for four types of risk measures:

- Semi-variance
- Value-at-Risk (VaR)

- Conditional Value-at-Risk (CVaR), aka Expected Shortfall
- Manipulation Proof Performance Measure (MPPM)

We will also explore the Aumann and Serrano Index of Riskiness, adapted to the performance of investments.

We will map the sensitivity of the adjustment to negative skewness and excess kurtosis by using a Cornish-Fisher transformation which allows one to generate distributions with the desired characteristics for a wide range of third and fourth moments.

3 – The Cornish-Fisher transformation

The Cornish-Fisher transformation and the proper way to use it are described in Maillard (2012). It consists of transforming the quantiles of the normal standard distribution z into quantiles of a new distribution Z allowing for a controlled skewness and kurtosis.

$$Z = z + (z^2 - 1)\frac{S}{6} + (z^3 - 3z)\frac{K}{24} - (2z^3 - 5z)\frac{S^2}{36}$$

In that expression, S and K are parameters which allow to control for skewness and kurtosis, but are not the actual values of the skewness and kurtosis of the transformed distribution. The value of the first four moments of the distribution are given in Appendix.

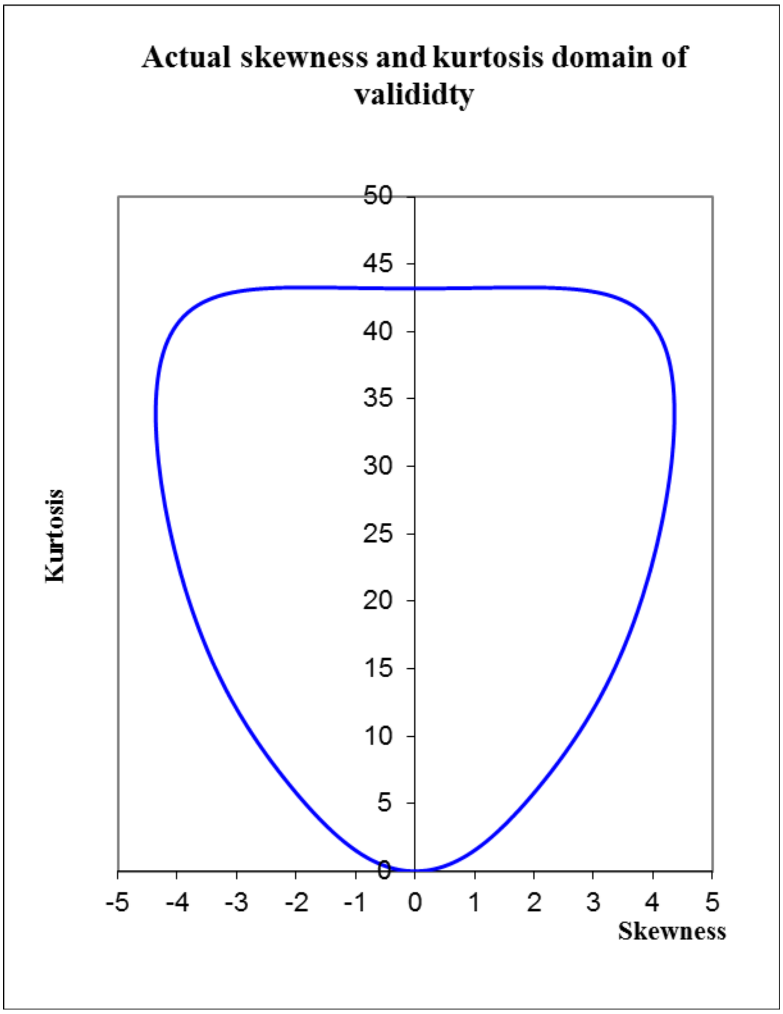
Note that to obtain a unitary variance of the transformed distribution, we have to divide the resulting quantiles by the square root of M2 given in appendix[2].

To obtain a distribution with zero mean, unitary variance, and desired skewness $\hat{S}$ and kurtosis $\hat{K}$, we have to reverse the expressions given in Appendix to obtain parameters S and K (a resolution along the response surface method is the object of a forthcoming paper[3]).

[2] We will in fact use in the modelling of the quantiles (49999) in our numerical applications, the following transformation : $Z' = \dfrac{z + (z^2 - 1)\frac{S}{6} + (z^3 - 3z)\frac{K}{24} - (2z^3 - 5z)\frac{S^2}{36}}{\sqrt{1 + \frac{1}{96}K^2 + \frac{25}{1296}S^4 - \frac{1}{36}KS^2}}$

An important caveat in the use of the transformation is that it should correspond to a true probability law, i.e. adding to one (which is respected) and non-negativity, meaning here that quantiles are uniformly increasing. This leads to a domain of validity which is convenient for most asset and portfolio returns.

Chart 1



This domain of validity is much larger than one obtained using Gramm-Charlier transformations of the probability law.

[3] "Computation of the corrected Cornish–Fisher expansion using the response surface methodology: application to ***VaR*** and ***CvaR*",** Charles-Olivier Amédée-Manesme, Fabrice Barthélémy and Didier Maillard, Annals of Operations Research (2018)

4 – Adjustment with semi variance

If Φ is the probability distribution of mean 0 and variance 1, the semi-variance will be equal to:

$$SV = \int_{-\infty}^{0} \varepsilon^2 \Phi(\varepsilon) d\varepsilon$$

If the distribution is Gaussian, it is obvious that:

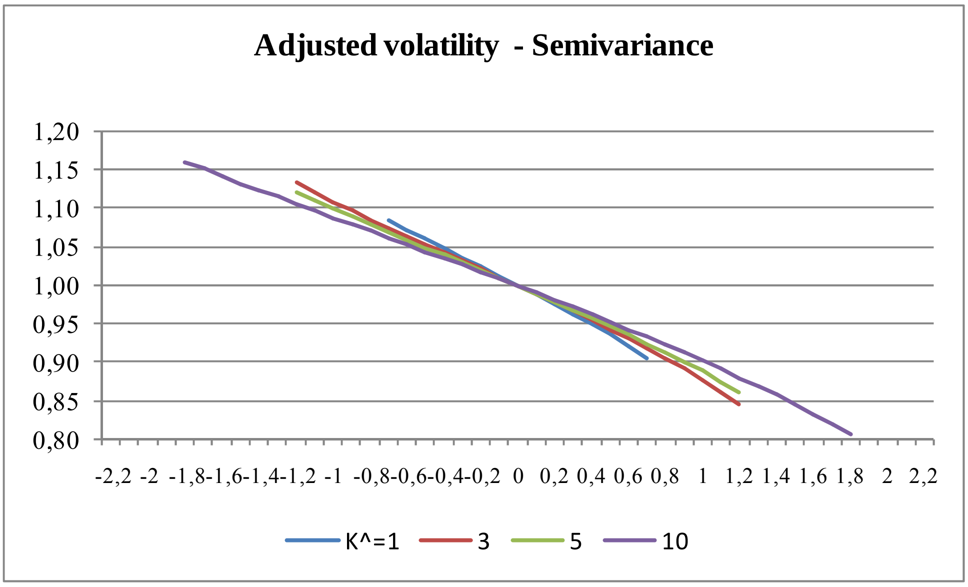
$$SV = \frac{1}{2} V = \frac{1}{2}$$

For a non-Gaussian distribution, we will compute SV as above, and correct the volatility of the TRaSR as:

$$\sigma_a = \sigma \frac{\sqrt{SV}}{\sqrt{1/2}} = \sigma\sqrt{2SV}$$

The chart below gives the adjusted volatility (divided by the actual volatility) as a function of skewness for some levels of kurtosis.

Chart 2



Two main findings appear. The first one is that the dependence on kurtosis is small. It is nil when there is no skewness (which is expected). The second one is that the dependency on skewness is nearly linear and that one point of negative skewness increases the tail-risk adjusted volatility by around 10 %, and thus decreases the Sharpe ratio by a near identical percentage.

For further work, we could explore the denominator of Sortino ratios at various thresholds

$$D = \int_{-\infty}^{\tau} \varepsilon^2 \Phi(\varepsilon) d\varepsilon$$

And transform them into tail-risk adjusted volatilities.

4 – Adjustment with Value-at-Risk

For any distribution (centred) with quantiles *Z*, value-at-risk (centred and reduced) at confidence level 1-*α* is:

$$VaR_{1-\alpha} = -\sigma Z_\alpha$$

For a Gaussian distribution, value-at-risk (centred) at confidence level 1-*α* is:

$$VaR_{1-\alpha} = \sigma v_\alpha = -\sigma z_\alpha = -\sigma N^{-1}(\alpha)$$

We will thus define the tail-risk adjusted volatility as the volatility at which the Gaussian distribution gives the same VaR as the actual Var.

$$-\sigma_a z_\alpha = -\sigma Z_\alpha$$

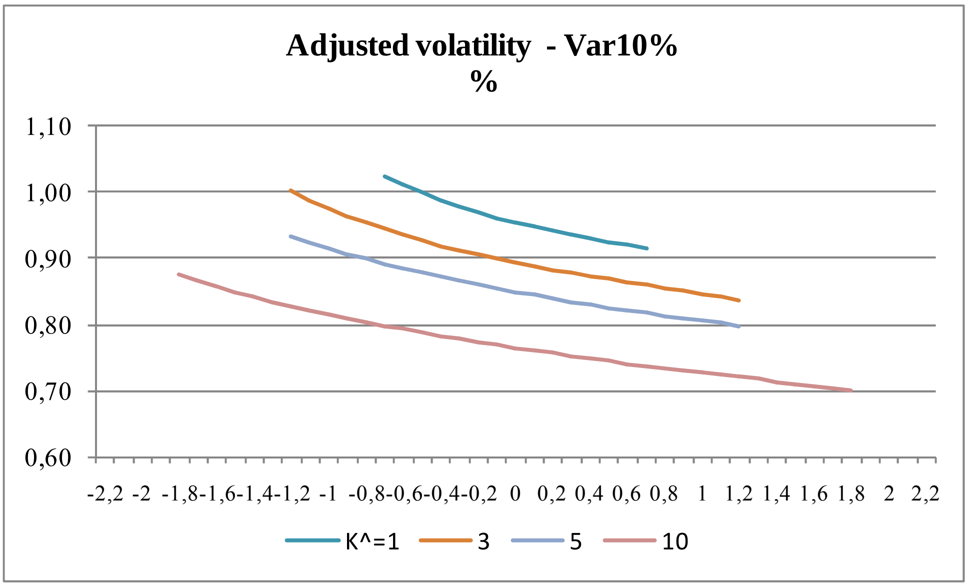
$$\frac{\sigma_a}{\sigma} = \frac{Z_\alpha}{z_\alpha}$$

$$-Z_\alpha = v_\alpha + (1 - v_\alpha^2)\frac{S}{6} + (5v_\alpha - 2v_\alpha^3)\frac{S^2}{36} + (v_\alpha^3 - 3v_\alpha)\frac{K}{24}$$

Value-at-risk can be computed with this formula or read in the numerical projections of quantiles (in both cases, with the correction referred in a previous footnote).

The charts below give the adjustment of volatilities for various levels of skewness and kurtosis and commonly used levels of Value-at-risk (α).

Chart 3a



For a 10% threshold, we see that negative skewness has a cost, but it is reduced when associated with a high kurtosis. As a matter of fact, by reducing the probability of median sized negative events, high kurtosis has the effect of reducing the position of the value-at-risk. This points to a shortcoming of the Value-at-Risk measure (which does not appear in the case of CVaR).

Chart 3b

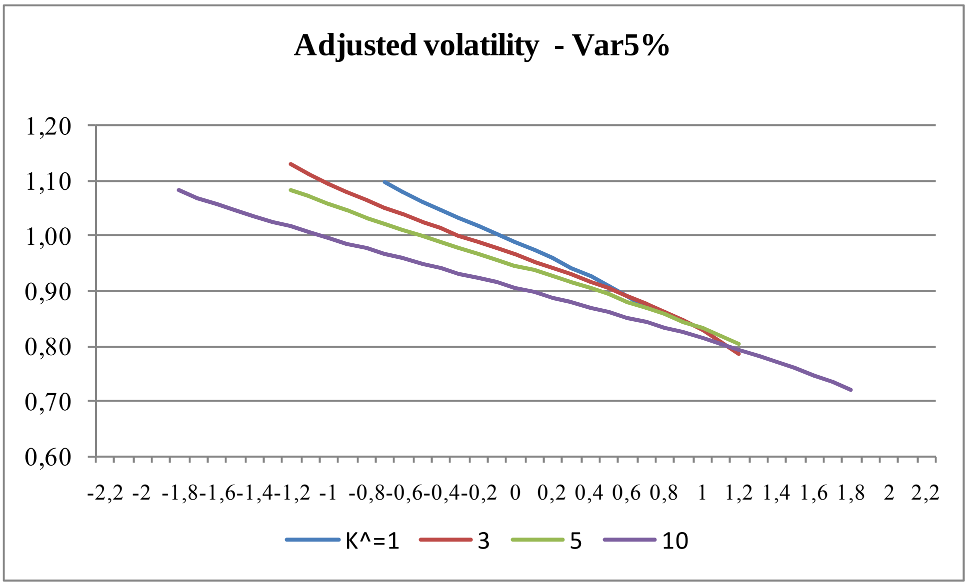
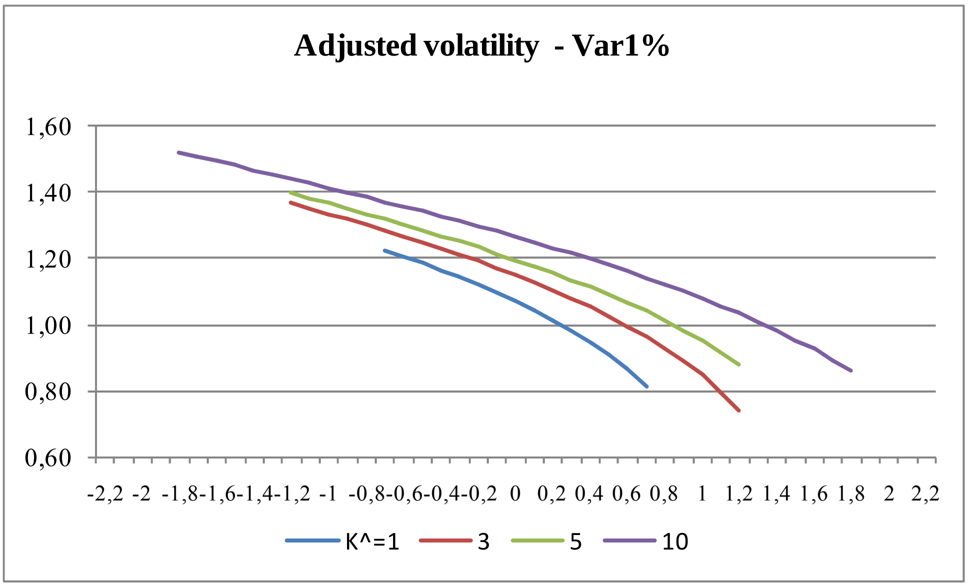


Chart 3c



5 – Adjustment with Conditional Value-at-Risk

As in the case of VaR, there is an analytical expression of CVaR (Maillard, 2012) resulting from the Cornish-Fisher transformation.

$$CvaR_{1-\alpha} = Y_\alpha = y_\alpha - \frac{S}{6} v_\alpha y_\alpha + \frac{S^2}{36}(y_\alpha - 2y_\alpha v_\alpha^2) + \frac{K}{24}(-y_\alpha + y_\alpha v_\alpha^2)$$

$$y_\alpha = \frac{1}{\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{z_\alpha^2}{2}}$$

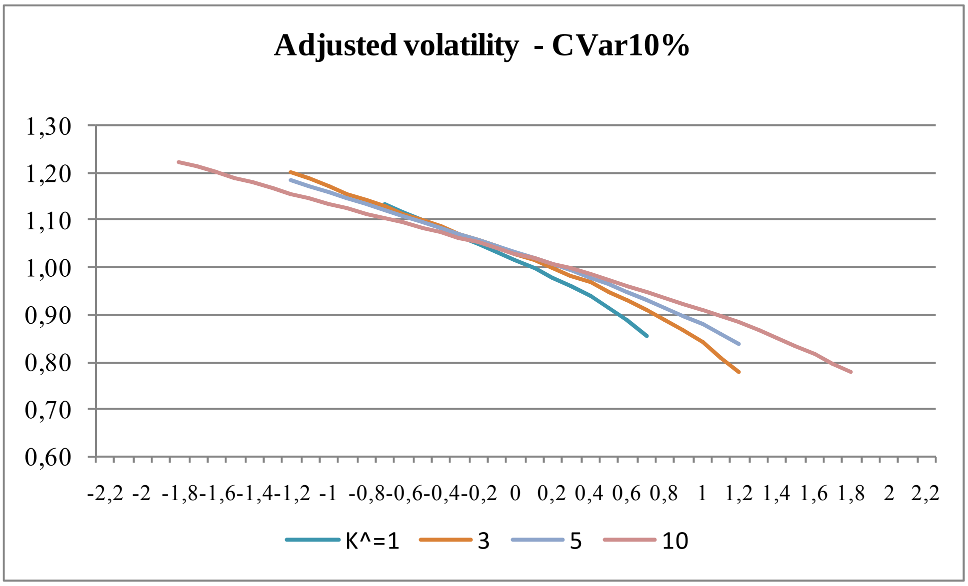
y_α is the CvaR of the normal standard distribution.

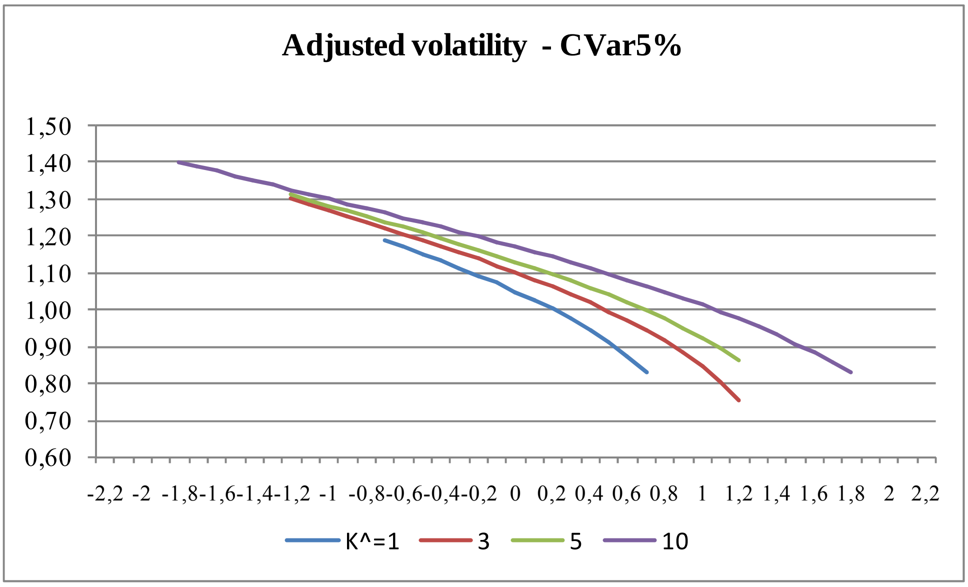
As in the case of VaR, we will define the Tail-risk adjusted volatility by:

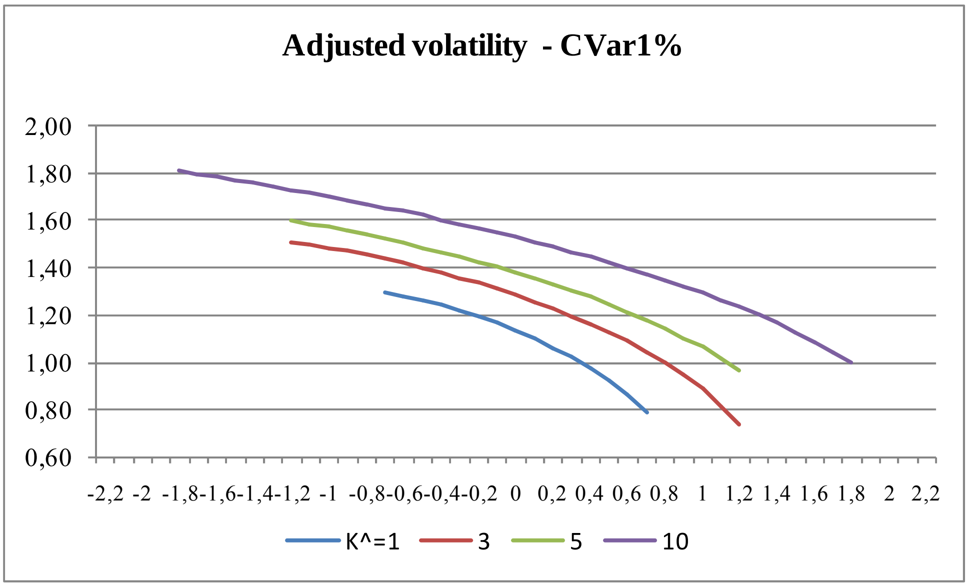
$$\frac{\sigma_\alpha}{\sigma} = \frac{Y_\alpha}{y_\alpha}$$

For all thresholds, the adjusted volatility increases with negative skewness.

Chart 4a







6 – Adjustment with Manipulation Performance Measure

The use of Goeztman (&al.) performance measure to assess the impact of tail risk has been extensively described in Maillard (2013b).

As a way to counter all type of manipulation, they propose a Manipulation-Proof Performance measure (MPPM) which writes[4]:

[4] We stick to the authors' notations, except that we substitute γ for ρ. γ is the usual symbol for denoting a relative risk aversion (RRA), and the parameter in the MPPM measure may be assimilated to a RRA (in the Goetzmann & alii's paper, γ is used for another purpose).

$$\hat{\Theta} = \frac{1}{(1-\gamma)\Delta t} \ln\left(\frac{1}{T} \sum_{t=1}^{T} \left[\frac{1+r_t}{1+r_{ft}} \right]^{1-\gamma} \right)$$

T is the number of observations, γ is a parameter related to risk aversion, r_{ft} the risk-free rate for period t (assuming such thing still exists…), and Δt the length of the period (in years) on which the return is recorded. r_t is the return of the fund during period t. Implicitly, the risk-free rate acts as a benchmark against which the performance is measured. The ratio in the formula is one plus a geometric excess return x_t. The exponentiation by 1- γ of the relative performance is there to take risk into account.

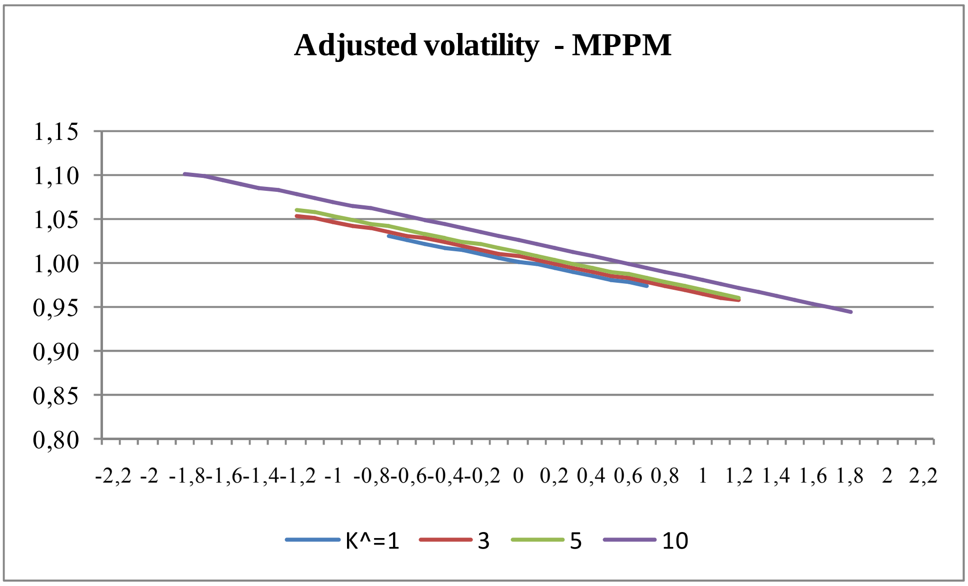
As the authors state, the MPPM is very close to an expected utility, of the power or CRRA (Constant Relative Risk Aversion) form (with RRA equal to γ), of an end-of-period wealth, which an investor would like to optimize. By taking the logarithm and dividing by the length of the period and one minus the risk aversion parameter, they ensure that the measure is equivalent to an equivalent-certain (continuous) rate of return.

The MPPM is very close to the Morningstar Risk-Adjusted Return (MRAR) that this firm uses to compare the performance of various funds (the return computed by Morningstar is in traditional and not continuous form).

The adjusted volatility is the volatility of a Gaussian distribution leading to the same MPPM measure than the non-Gaussian one.

The chart below is computed for γ = 5 and σ = 6 % (usual for monthly returns of a risky asset).

Chart 5



The adjustment depends on the risk aversion parameter used but also on the level of volatility. Kurtosis does not impact much and the relationship with skewness is nearly linear.

7 – Aumann and Serrano-s Index of Riskiness

In a paper (Aumann & Serrano, 2008), Aumann and Serrano propose an economic index of riskiness. For a gamble or a lottery whose pay-off is a random value *g*, the index of riskiness *R* is such that:

$$E(e^{-g/R}) = 1$$

Translated into the investment field, the gamble consists of exchanging the risk free rate r_f against the return of a risky asset, so that the pay-off is:

$$g = \mu - r_f + \sigma\varepsilon$$
$$E(\varepsilon) = 0 \quad E(\varepsilon^2) = 1$$

Assuming that the random part of the pay-off has no tail risk, i.e. that ε is normal standard, the index of riskiness verifies:

$$E(e^{-(\mu-r_f+\sigma\varepsilon)/R}) = e^{-(\mu-r_f)/R}E(e^{-\sigma\varepsilon/R}) = e^{-(\mu-r_f)/R}e^{\sigma^2/2R^2} = 1$$
$$(\mu-r_f)/R = \sigma^2/2R^2$$
$$R = \frac{\sigma^2}{2(\mu-r_f)}$$

For a non-Gaussian distribution, we will compute the Tail-risk adjusted volatility, after having computed the Economic Index of Riskiness of the distribution *R*, by

$$\sigma_a = \sqrt{2(\mu-r_f)R}$$

A first exploration, on the assumption of an expected excess return of 4%, a volatility of 20 %, gives an IER of .5 in the Gaussian case.

Increasing excess kurtosis by 3 increases the IER to .52, and increases the Tail-risk adjusted volatility by a factor of 1.02.

From that kurtosis base, adding negative skewness of -1 increases the IER to .5775, and the Tail-risk adjusted skewness by a factor of 1.0745. Conversely, positive skewness of 1 puts the IER to .445, and the volatility is adjusted by a factor of .9425.

8 - Conclusions

We have been able to propose several ways of correcting the Sharpe ratio, and the corresponding volatility, to take into account the higher moments of the distribution, i.e. tail risk.

Some of the measures analysed display an explicit parameter which may be linked to the investor's risk aversion: the thresholds of the VaR and CVaR (and the threshold of the Sortino ratio), the relative risk aversion parameter of the manipulation-proof performance measure.

Others do not display such explicit parameter: semi-variance and the Aumann-Serrano risk measure (though there is some resemblance with an absolute risk aversion equal to 1…).

All in all, there is a wide range of ways to correct the Sharpe Ratio for the existence of tail risk, with keeping the scaling of the measure of risk. The investor may choose its preferred one.

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Appendix

Moments of the Cornish-Fisher distribution

$$M1 = 0$$

$$M2 = 1 + \frac{1}{96}K^2 + \frac{25}{1296}S^4 - \frac{1}{36}KS^2$$

$$M3 = S - \frac{76}{216}S^3 + \frac{85}{1296}S^5 + \frac{1}{4}KS - \frac{13}{144}KS^3 + \frac{1}{32}K^2 S$$

$$M4 = 3 + K + \frac{7}{16}K^2 + \frac{3}{32}K^3 + \frac{31}{3072}K^4 - \frac{7}{216}S^4 - \frac{25}{486}S^6 + \frac{21665}{559872}S^8 - \frac{7}{12}KS^2 + \frac{113}{432}KS^4 - \frac{5155}{46656}KS^6 - \frac{7}{24}K^2S^2 + \frac{2455}{20736}K^2S^4 - \frac{65}{1152}K^3S^2$$

This leads to the actual values of skewness and (excess) kurtosis:

$$\hat{S} = \frac{M3}{M2^{1.5}} = \frac{S - \frac{76}{216}S^3 + \frac{85}{1296}S^5 + \frac{1}{4}KS - \frac{13}{144}KS^3 + \frac{1}{32}K^2 S}{\left(1 + \frac{1}{96}K^2 + \frac{25}{1296}S^4 - \frac{1}{36}KS^2\right)^{1.5}}$$

$$\hat{K} = \frac{M4}{M2^2} - 3 = \frac{\left[\begin{array}{l} 3 + K + \frac{7}{16}K^2 + \frac{3}{32}K^3 + \frac{31}{3072}K^4 - \frac{7}{216}S^4 - \frac{25}{486}S^6 + \frac{21665}{559872}S^8 - \frac{7}{12}KS^2 \\ + \frac{113}{432}KS^4 - \frac{5155}{46656}KS^6 - \frac{7}{24}K^2S^2 + \frac{2455}{20736}K^2S^4 - \frac{65}{1152}K^3S^2 \end{array}\right]}{\left(1 + \frac{1}{96}K^2 + \frac{25}{1296}S^4 - \frac{1}{36}KS^2\right)^2} - 3$$

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