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About the author

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Abstract

Investors around the world face in principle the same investment performance opportunity on the Foreign Exchanges, as they do on the capital markets. This principle can be demonstrated to hold only if the logarithmic measure is used in the calculation of the exchange rate returns and not the conventional arithmetic measure. Only then a situation of equal opportunity may arise within the standard mean/variance portfolio optimisation framework.

In mathematical terms, the logarithmic norm ensures that the return/risk opportunity space is defined in a consistent way, in the sense that it complies with the fundamental Euclidean properties of length and distance. If the algebraic rules are being applied regardless in non-Euclidean space, undue situations arise that can be shown to be incompatible with the equal opportunity principle.

Two domains in the currency literature are reviewed where this mistake is recurrently committed, one concerning the Uncovered Interest Rate Parity and the other the International Capital Asset Pricing Model.

Keywords: Siegel paradox, Euclidean distance function

JEL codes: C65, F31

1. Introduction

Unlike on the capital markets, the idea that investors worldwide have the same chances to win or lose on the Foreign Exchanges is not instantly obvious. Indeed it is not obvious at first glance how investors, who don't share a same currency perspective and therefore not perceive exchange rate movements identically, would seize and benefit from a same performance opportunity. Yet this complication is easily resolved by applying the appropriate conversion operators to switch from one currency perspective to another, as is laid out by Solnik (1974) and discussed in section 2.

More puzzling is the issue known as Siegel's (1972) paradox. When two investors exchange a sum of money in their respective currencies over a certain period, the arithmetic returns on that exchange -which is necessarily a gain for one and a loss for the other- do not add up to zero, as is made apparent in equation (1) below. Given this asymmetry in returns, it is not evident if investors with different currency perspectives who rely on the arithmetic measure in detecting opportunity, would assess a given market situation identically. From there it is not obvious whether they have strictly equal chances.

The Siegel paradox has provoked much debate over the last few decades. Renowned economists like Black (1989, 1995) and Krugman (1981) claim it to have real implications for currency investing, whereas others like Adler and Prasad (1992) waive it off as a trivial mathematical inconvenience. The purpose of this paper is to help settle this debate. It is demonstrated that, in the first place, the paradox is a mere virtual issue indeed, inherent to the arithmetic return measure, and that the 'real implications' cede as soon as the logarithmic measure is used instead. In the second place, it is shown that only with the logarithmic measure the principle of equal opportunity can be proven to hold. The two results put together make clear that there is a compatibility problem between the proclaimed real implications of Siegel and the principle of equal opportunity.

This short paper is organized as follows. Section 2 covers the mathematics; it formulates the inconsistency problem underlying the Siegel paradox and it gives formal proof of the equal opportunity principle. Section 3 reviews two domains in the currency literature where the paradox has given rise to a series of misunderstandings. Conflicting viewpoints are discussed that can be traced back to be in essence related to a choice of return measure. Section 4 concludes.

2. Measuring currency return and investment opportunity

Let us start by restating the paradox named after Siegel (1972). Let S_t be the spot rate of a currency stated in a numeraire at time *t*, then $S_t^* = 1/S_t$ is the inverse, the spot rate of the numeraire stated in the currency. The sum of the arithmetic returns (R^{ar}) over one period measured from the two perspectives is not zero, but is strictly positive, as is made apparent in equation (1):

$$R_t^{ar} + R_t^{ar*} = \left(\frac{S_t}{S_{t-1}} - 1\right) + \left(\frac{1/S_t}{1/S_{t-1}} - 1\right) = \frac{(S_t - S_{t-1})^2}{S_t S_{t-1}} > 0 \quad \text{if } S_t \neq S_{t-1} \tag{1}$$

Black (1989, 1990 and 1995) has formulated the paradox in terms of return expectations as opposed to realised returns. On an imaginary barter exchange, he writes, one apple is trading for one orange at *t*, while at *t+1* either one apple is trading for two oranges or one orange for two apples with equal probability. The expected arithmetic return at *t* of exchanging a fruit is again strictly positive:

$$E\left[R_t^{ar}\right] = 0.5 \cdot (2-1) + 0.5 \cdot \left(1/2 - 1\right) = 0.25 \tag{2}$$

It is indeed paradoxical that there seems to be a pure numerical benefit in exchanging goods or money. The illusion can be easily lifted though by introducing alternative measures of return. If the Fisher measure (R^F) were applied for instance, the sum would be exactly zero:

$$R_t^F + R_t^{F*} = \frac{S_t - S_{t-1}}{\sqrt{S_t \cdot S_{t-1}}} + \frac{1/S_t - 1/S_{t-1}}{\sqrt{1/S_t \cdot 1/S_{t-1}}} = \frac{(S_t - S_{t-1}) + (S_{t-1} - S_t)}{\sqrt{S_t \cdot S_{t-1}}} = 0, \tag{3}$$

or alternatively, if the logarithmic return (R^{log}) measure were applied:

$$R_t^{log} + R_t^{log*} = \ln\left(\frac{S_t}{S_{t-1}}\right) + \ln\left(\frac{S_{t-1}}{S_t}\right) = 0. \tag{4}$$

Those two measures are said to be additive across the perspectives, a quality the arithmetic measure doesn't possess for a pure algebraic reason known as Jensen's inequality. Fisher (1922) had initially introduced his measure exactly for this purpose, to avoid elementary consistency problems.

In the literature many have alerted for the inconsistency problem in the exchange rate return calculation, insisting that it has no real implications for currency investing. The fervent belief in the contrary -this strand of literature is covered in next section- shows that the arguments that have been brought forward are manifestly unconvincing. That may indeed be the case. McCulloch (1975), for instance, misses the point when he states that the Siegel paradox has no real implications because the discrepancies are too small in practice to be a real obstacle. Engel (1984) has unduly held ‘*money illusion on the part of market participants*’ responsible for the belief in a real benefit. The standard reference text books[1] are not convincing either, remaining conveniently vague on the issue.

I introduce a more persuasive way to formulate the inconsistency problem underlying the Siegel paradox is. It is to say that the arithmetic norm doesn’t respect the Euclidean properties of length and distance, when applied on currencies. Let return be a measure for length, then, the variance of the returns measures the distance between currencies. It can easily be shown that the three Euclidean conditions for a distance function, which are formulated below, are not met when the arithmetic norm is used.

Definition the function $d: X \cdot X \to \Re$ is a Euclidean distance function if the following three conditions hold

$$\begin{array}{lll} (i) & \textit{postive definiteness} & d(x,y) \geq 0 \;\; \textit{and} \;\; d(x,y) = 0 \quad \textit{if} \;\; x = y \\ (ii) & \textit{symmetry} & d(x,y) = d(y,x) \\ (iii) & \textit{triangle inequality} & d(x,z) \leq d(x,y) + d(y,z) \end{array} \tag{5}$$

In particular, the arithmetic norm fails on the property of symmetry. The variances of a spot rate *S* measured from the two currency perspectives, denoted as σ^2 and σ^{2*}, are not identical:

$$\sigma^2 = \sum_t \frac{(S_t - S_{t-1})^2}{S_{t-1}^2} \Big/ T \;\; \neq \;\; \sum_t \frac{(S_{t-1} - S_t)^2}{S_t^2} \Big/ T = \sigma^{2*} \tag{6}$$

The reader may verify that the Fisher norm fails on the symmetry condition as well, as Fisher returns are not time additive. Yet the logarithmic norm fully complies.

With the intention to help settle the debate on the Siegel paradox, I rephrase what I consider to be the crux of the issue: the Euclidean properties are violated by the arithmetic norm, and therefore the conditions are not met to apply the usual rules of algebra. Erroneous calculations are being made and faulty conclusions are drawn, if algebraic operations are carried out regardless in non-Euclidean space. Adler and Prasad (1992) have said as much, although their

[1] See Obstfeld and Rogoff (1996) pp.587-588 or Sarno and Taylor (2009) pp.36-37.

phrasing is somewhat incomprehensible. In their abstract they write: "*Jensen's inequality is what makes the choice of measurement currency irrelevant*."

In the same article Adler and Prasad make the crucial link between the choice of return measure and the principle of equal investment opportunity, on page 29: "*[Jensen's inequality] guarantees that the portfolio weights of all assets, whether chosen for speculation or hedging, remain constant across diverse investors [with different currency perspectives]*." The importance of making this link seems to have been ignored. They make reference to Solnik's lemma (1974), which states that the traditional mean/variance portfolio optimisation problem is indifferent as to the base currency in which it is formulated. The lemma and its proof are given below.

Lemma Let $\wp_i$ be an unconstrained Markowitz optimisation problem defined on *N* currencies stated in reference currency *i*.

$$\wp_i : \text{maximise } u(x_i) = R_i^T x_i - \lambda \cdot x_i^T V_i x_i \tag{7}$$

where *u(.)* denotes the utility function, R_i is a given vector containing *N-1* exchange rate return expectations, x_i contains the *N-1* optimal holdings, λ is the aversion to risk and V_i is the (*N-1*)-dimensional covariance matrix. Then,

$$\forall_{j \in N} : \boldsymbol{u}(\boldsymbol{x}_j) = \boldsymbol{u}(\boldsymbol{x}_i)$$

Proof Let H_{ij} be the conversion operator from currency *i* into currency *j*.

$$\mathrm{H_{ij}} = \begin{array}{c} \begin{bmatrix} 1 & 0 & -1 & .. & 0 \\ : & \ddots & : & .. & : \\ 0 & 0 & -1 & .. & 0 \\ 0 & 0 & -1 & \ddots & 0 \\ 0 & 0 & -1 & .. & 1 \end{bmatrix} \\ \begin{array}{cc} \Downarrow & \\ \mathrm{j^{th}\,column} & \end{array} \end{array} \quad (\mathrm{N}-1)\mathrm{x}(\mathrm{N}-1) \tag{8}$$

One can easily verify that the conversion of the return vector *R* and of the optimised portfolio *x* is effectuated by a pre-multiplication, and the conversion of the covariance matrix *V* by a pre- and post multiplication, as follows:

$$R_j = H_{ij} \cdot R_i,\ x_j = H_{ij} \cdot x_i \text{ and } V_j = H_{ij} \cdot V_i \cdot H_{ij}^T \tag{9}$$

The utility of the portfolio in currency perspective *j* is

$$u(x_j) = R_i^T \cdot H_{ij}^T \cdot H_{ij} \cdot x_i - \lambda \cdot x_i^T H_{ij}^T \cdot H_{ij} \cdot V_i \cdot H_{ij}^T \cdot H_{ij} \cdot x_i = u(x_i) \qquad (10)$$

since $H_{ij}^T \cdot H_{ij} = I_{N-1}$ **Q.E.D.**

Note that investors do not take position identically while aiming for the same performance opportunity, formally $x_j \neq x_i$ while $u(x_j) = u(x_i)$. Note more importantly, that the lemma only holds if the returns are calculated with the logarithmic measure. The linear conversion relationships, given in equations (9), would not hold with arithmetic returns. Moreover, note that the covariance matrix *V* must be positive definite and therefore symmetrical (first Euclidean condition), for the problem to be well defined and have a unique solution. However, recall from equation (6) that any historical covariance matrix measured with the arithmetic norm is not symmetrical and wouldn't be valid.

3. Debate over the Siegel paradox

Siegel (1972) himself had initially brought the paradox forward as a mere curiosity, yet in a follow-up paper published in 1975 he disowns the belief that it has any real implications for currency investment. Despite his declaration, a strand of literature had erupted that conjectures the implications to be real. The conjecture is being upheld in two domains: in the interpretation of the Uncovered Interest Rate Parity (*UIP*), and in connection with the International Capital Asset Pricing Model (*I-CAPM*). A review is given in both domains.

3.1. The Uncovered Interest Rate Parity

The Covered Interest Rate Parity states that the ratio of the *k*-period forward- (*F*) versus the spot (*S*) exchange rate between any two currencies is identical to the ratio of the respective interest rates (*i* and *i**) over *k*. Algebraically,

$$\frac{F_t^k}{S_t} = \frac{1+i_t}{1+i_t^*} \qquad (11)$$

Since all variables are observed and tradable at *t*, it is evident that any diversion from the identity equation would be arbitraged away quickly. The *UIP* states that with regard to market efficiency theory, the expected spot exchange rate at *t*+*k* should convert towards the interest rate ratio over *k* :

$$\frac{E\left[S_{t+k}\right]}{S_t} = \frac{1+i_t}{1+i_t^*} \tag{12}$$

Many have tried to verify whether the parity holds empirically. Typically it is tested whether the forward rate is an unbiased estimator of the future spot rate. Those tests are haunted by the Siegel problem, in that the forward rate cannot be an unbiased estimator in both numeraires simultaneously.

The null-hypothesis $H_0 : F_t^k = E\left[S_{t+k}\right]$ *and* $\dfrac{1}{F_t^k} = E\left[\dfrac{1}{S_{t+k}}\right]$ (13)

is violated because $E\left[\dfrac{1}{S_{t+k}}\right] \neq \dfrac{1}{E\left[S_{t+k}\right]}$.

Engel (1984) proposes a test set-up in a way that the Siegel problem disappears. Instead of testing on the nominal spot rate returns, he suggests to test on real spot rate returns which are corrected for the evolution in prices of tradable goods (*P*). Since he assumes that the Purchase Power Parity continuously holds, i.e. $P_{t+k}^* = P_{t+k} \cdot S_{t+1}$, the null-hypothesis of the bias test is respected both ways.

$$\text{If } H_0 : E\left[\frac{F_t^k - S_{t+k}}{P_{t+k}}\right] = 0 \text{ then } E\left[\frac{{}^{1}\!/\!{}_{F_t^k} - {}^{1}\!/\!{}_{S_{t+k}}}{P_{t+k}^*}\right] = 0 \tag{14}$$

The virtue of this test set-up is that the return measure which is used in the statistical inference respects the Euclidean properties. It is not pertinent though to draw conclusions on the price behaviour of currencies with respect to the price level of tradable goods. Engel (1984) himself is ambivalent on this point. On the one hand he propagates, on page 307, that '*there has been little recognition of the implicit need to choose a standard by which to measure profits*.' On the other hand he claims, in his abstract, that his tests on real- rather than on nominal returns eliminate the effects of money illusion, whereas in reality his tests provide no basis whatsoever to conclude anything on money illusion.

Obstfeld and Rogoff (1996) are careful, while discussing Engel's proposition, to stress that it resolves a measurement issue, yet they do comment that '*risk-neutral investors should care about real returns, not nominal returns*'. Such comments give new ground to perpetuate the ambivalence. Many unduly call the Siegel problem '*a nominal phenomenon with no real implications*', as does Beenstock (1985). Kemp and Sinn (1990, 2000) go as far as to claim that the paradox creates tradable and profitable arbitrage opportunities. They recommend a closure of the forward markets in order to stop what-they-call socially useless speculation!

3.2. The International CAPM

The *I-CAPM*, developed by Solnik (1974), is the generalisation of the fundamental equity pricing model of Sharpe (1964) that was originally destined to local markets, towards an international setting. Inherently currencies become part of the equity pricing process. Solnik argues that the Market Portfolio that investors are inclined to hold in general price equilibrium is unique for all, as in the original model. If currency prices were perfectly uncorrelated to equity prices, investors would in equilibrium fully hedge all their equity positions. If not, investors would install an accorded partial hedge. Thus, the default optimal hedge ratio is in principle 100 percent for all investors, according to Solnik and McLeavey (2008), though it may differ depending on the view on the covariance structure between currency and equity prices.

Black (1989, 1995) openly disputes this point of view. He calls on the Siegel paradox to defend that investors would always install a partial currency hedge in market equilibrium disregarding the correlation structure with the assets in the portfolio. The fact that the aggregate return expectation of currencies is strictly positive –in the arithmetic measure that is- there is, according to Black, a mutual benefit in systematically holding foreign currency. All investors would benefit from leaving a part of their international equity positions non-hedged.

Black writes in 1989:"*Siegel's paradox makes investors want significant amounts of exchange rate risk in their portfolios. It also makes investors prefer a world with more exchange rate risk to a similar world with less exchange rate risk*.", and in 1995: "[*The Siegel paradox*] *isn't a mathematical trick. It's real and it means that investors generally want to hedge less than 100 percent of their foreign investments*."

Solnik and Black never settled their debate and it is surprising how little Black got challenged by others. On the contrary, it seems that many think along the same lines. Krugman (1981) argues that, besides the return benefit, there also is risk diversification benefit from holding foreign currency cash, making a vague reference to the old all-eggs-in-one-basket argument. He doesn't give a demonstration of his argument though. The trouble is that home cash is assumed to be risk-free and it is inconceivable to build a diversified currency portfolio that beats zero risk. He dismisses the issue by supposing that the costs of maintaining an international portfolio probably outweigh the benefits anyway.

It is difficult to reconcile Black's untenable position on currencies with the elegant equilibrium theory he has developed together with Litterman (1992). They develop the argument that the Market Portfolio held in price equilibrium within a given investment universe, confronted against the covariance structure between the assets, reveals the equilibrium risk premiums that investors expect from holding the assets. The direct relation between the (optimal) portfolio holdings and the implied return expectations can be made explicit. In the Markowitz (1952) framework the relation is derived by setting the first order condition of the objective function, given in (7), to zero, obtaining:

$$\lambda V x = R \tag{15}$$

where x contains the Market Portfolio, V the covariance matrix,

R the equilibrium risk premiums, and is λ the aversion to risk.

Via Black and Litterman's (1992) theory Black's standpoint can be shown to be less plausible than Solnik's standpoint. I give a demonstration making use of the investment example that Black and Litterman had used in their article on seven major capital markets, given in Table 1 below. The Market Portfolio (MP), taken from their Table VII and displayed in grey, confronted against the covariance matrix, reconstituted from their Tables I and II, corresponds exactly with the risk premiums (R) given in the rightmost column. They are 1 percent per annum on average for bonds and 6 percent for equity. These return levels are indeed observed empirically over long periods of time and are in line with the return-to-risk profiles that are usually associated to these asset classes.

However, if the partial currency hedge were applied, as they suggest in the article, onto the Market Portfolio, the portfolio would include long positions on foreign (non-US) currencies, as in the grey column in the Table. The implied risk premiums matching with those holdings are calculated to be all positive, of around 1 percent per year in the example. This is odd. It expresses the view that the dollar is set to depreciate by around 1 percent per annum with respect to the other currencies indefinitely in time!

Table 1 Black and Litterman's investment example

	co-	Equity							Bonds							Currency								
	variance	DEM	FRF	JPY	GBP	USD	CAD	AUD	DEM	FRF	JPY	GBP	USD	CAD	AUD	DEM	FRF	JPY	GBP	CAD	AUD	*MP*	*R*	
Equity	DEM	3.3%	2.1%	1.2%	1.9%	1.3%	1.1%	1.4%	0.2%	0.2%	0.1%	0.3%	0.2%	0.2%	0.2%	0.0%	0.1%	0.0%	0.0%	0.0%	0.0%	2.6%	4.5%	**6%**
	FRF	2.1%	4.9%	1.7%	2.7%	1.9%	2.0%	1.9%	0.2%	0.4%	0.2%	0.4%	0.2%	0.2%	0.0%	0.1%	0.2%	0.3%	0.1%	0.1%	0.2%	2.4%	6.4%	
	JPY	1.2%	1.7%	3.2%	1.6%	1.2%	1.1%	1.0%	0.1%	0.2%	0.4%	0.4%	0.1%	0.2%	0.1%	0.1%	0.1%	0.4%	0.1%	0.1%	0.1%	23.7%	6.1%	
	GBP	1.9%	2.7%	1.6%	6.1%	2.3%	2.5%	2.7%	0.2%	0.2%	0.1%	1.1%	0.2%	0.3%	0.2%	0.0%	0.1%	0.1%	0.2%	0.2%	0.2%	8.3%	8.0%	
	USD	1.3%	1.9%	1.2%	2.3%	2.6%	2.2%	1.7%	0.2%	0.2%	0.1%	0.4%	0.4%	0.4%	0.2%	0.1%	0.1%	0.0%	0.0%	0.2%	0.1%	29.7%	6.3%	
	CAD	1.1%	2.0%	1.1%	2.5%	2.2%	3.3%	2.4%	0.1%	0.0%	0.0%	0.5%	0.2%	0.3%	0.2%	0.1%	0.2%	0.1%	0.2%	0.3%	0.4%	1.6%	5.8%	
	AUD	1.4%	1.9%	1.0%	2.7%	1.7%	2.4%	4.8%	0.1%	0.1%	0.0%	0.4%	-0.1%	0.0%	0.4%	0.0%	0.1%	0.3%	0.4%	0.2%	0.6%	1.1%	5.0%	
Bonds	DEM	0.2%	0.2%	0.1%	0.2%	0.2%	0.1%	0.1%	0.2%	0.1%	0.1%	0.2%	0.2%	0.2%	0.0%	0.2%	0.2%	0.1%	0.1%	0.0%	0.0%	2.9%	0.7%	**1%**
	FRF	0.2%	0.4%	0.2%	0.2%	0.2%	0.0%	0.1%	0.1%	0.2%	0.1%	0.1%	0.1%	0.1%	0.0%	0.1%	0.1%	0.1%	0.0%	0.0%	0.0%	1.9%	0.7%	
	JPY	0.1%	0.2%	0.4%	0.1%	0.1%	0.0%	0.0%	0.1%	0.1%	0.4%	0.2%	0.1%	0.2%	0.1%	0.2%	0.2%	0.4%	0.2%	0.0%	0.1%	6.0%	1.0%	
	GBP	0.3%	0.4%	0.4%	1.1%	0.4%	0.5%	0.4%	0.2%	0.1%	0.2%	1.0%	0.2%	0.3%	0.1%	0.1%	0.1%	0.2%	0.3%	0.1%	0.1%	1.8%	1.8%	
	USD	0.2%	0.2%	0.1%	0.2%	0.4%	0.2%	-0.1%	0.2%	0.1%	0.1%	0.2%	0.5%	0.4%	0.1%	0.2%	0.2%	0.2%	0.1%	0.0%	0.0%	16.3%	1.2%	
	CAD	0.2%	0.2%	0.2%	0.3%	0.4%	0.3%	0.0%	0.2%	0.1%	0.2%	0.3%	0.4%	0.6%	0.1%	0.2%	0.2%	0.2%	0.2%	0.1%	0.0%	1.4%	1.4%	
	AUD	0.2%	0.0%	0.1%	0.2%	0.2%	0.2%	0.4%	0.0%	0.0%	0.1%	0.1%	0.1%	0.1%	0.3%	0.1%	0.1%	0.1%	0.1%	0.0%	0.1%	0.3%	0.7%	
Currency	DEM	0.0%	0.1%	0.1%	0.0%	0.1%	0.1%	0.0%	0.2%	0.1%	0.2%	0.1%	0.2%	0.2%	0.1%	1.5%	1.3%	0.9%	1.0%	0.1%	0.3%	1.1%	0.9%	**1%**
	FRF	0.1%	0.2%	0.1%	0.1%	0.1%	0.2%	0.1%	0.2%	0.1%	0.2%	0.1%	0.2%	0.2%	0.1%	1.3%	1.4%	0.9%	0.9%	0.1%	0.3%	0.9%	1.0%	
	JPY	0.0%	0.3%	0.4%	0.1%	0.0%	0.1%	0.3%	0.1%	0.1%	0.4%	0.2%	0.2%	0.2%	0.1%	0.9%	0.9%	1.5%	0.8%	0.0%	0.3%	5.9%	1.2%	
	GBP	0.0%	0.1%	0.1%	0.2%	0.0%	0.2%	0.4%	0.1%	0.0%	0.2%	0.3%	0.1%	0.2%	0.1%	1.0%	0.9%	0.8%	1.4%	0.1%	0.3%	2.0%	0.8%	
	CAD	0.0%	0.1%	0.1%	0.2%	0.2%	0.3%	0.2%	0.0%	0.0%	0.0%	0.1%	0.0%	0.1%	0.0%	0.1%	0.1%	0.0%	0.1%	0.2%	0.1%	0.6%	0.5%	
	AUD	0.0%	0.2%	0.1%	0.2%	0.1%	0.4%	0.6%	0.0%	0.0%	0.1%	0.1%	0.0%	0.0%	0.1%	0.3%	0.3%	0.3%	0.3%	0.1%	1.1%	0.3%	0.6%	

Source: Statistics as measured by Black and Litterman (1992) on market data between 01/1975 and 08/1991, and printed in their Tables I, II and VII. The seven markets included are, in order, Germany, France, Japan, Great-Britain, the United States, Canada and Australia.

Black's standpoint that the foreign currency holdings are somehow justified by 'the mutually perceived positive returns' cannot be made fit in the algebraic equation. His standpoint is in contradiction with his own equilibrium theory. The only way to make non-zero currency holdings in the Market Portfolio match with strictly zero currency risk premiums, in line with market efficiency theory, would be to drop the property of symmetry in the covariance matrix. Only in that situation, in which the Euclidean properties are again flagrantly violated, could the match be technically achieved. The absurdity of Black's standpoint would become instantly clear.

Solnik's standpoint is much easier to justify. Since it is inconceivable to have perpetually diverging exchange rates, their equilibrium risk premiums must all be zero per definition. In the absence of correlation with the asset classes, the currency exposures that match must be zero as well. If correlations do exist with currencies, non-zero currency exposures corresponding with non-zero premiums can roll out of the Black & Litterman equation. In that case the premiums reflect indirectly the return expectations for other asset classes to which currency is related.

Ironically the choice of hedge that Black and Litterman had made in their investment example can be justified with Solnik's argument. The correlations of the currencies with equities and bonds are generally positive, as can be seen in the covariance matrix *V*, directly implying that the US dollar is negatively correlated to the equity and bond price levels. In that arguably untenable situation the expected return on the dollar is negative and consistent with that it is optimal to short the dollar.

The debate on international capital price equilibrium models remains unsettled in the literature. Many questions remain unresolved, most importantly the question how to optimally manage currency risk in an internationally invested portfolio. Both Black's and Solnik's position are being defended side by side, as do Campbell, Serfaty-de Medeiros and Viceira (2010). They explicitly mention both the Siegel paradox ànd the existence of correlation as rational reasons why to systematically maintain foreign cash positions in an international portfolio.

4. Conclusion

It is fair to say that the inconsistency problem in the exchange rate return calculations, known as the Siegel paradox, has conditioned the way thought and theory has been developed on currencies over the last few decades. The relatively minor geometrical issue has been a major source of confusion. Today the flaw in applying the conventional arithmetic return measure on currencies is still not fully recognised. The simple mathematical demonstration given in this paper may help coming to terms with the issue. It makes evident that the only adequate measure for currencies, at least in a portfolio management context, is the logarithmic measure.

Using the logarithmic instead of the arithmetic return measure on currencies paves the way towards a mean-variance optimisation setting where all market participants have in principle identical performance opportunity. It diffuses the illusion of an arbitrage opportunity in any money or goods exchange. The realised- as well as the expected return of an exchange is strictly zero when summed over the exchanging parties. That fact undoes the idea that international investors share a mutual benefit in holding foreign currencies.

In addition, the risk structures, captured by covariance matrices, can exactly match between different base perspectives only if the logarithmic norm is used. Given an expectation of return and risk expressed in logs, the corresponding mean/variance optimal portfolio is indifferent to the base perspective in which is optimised. The expectations as well as the optimised portfolio can be converted from one perspective into another with a standard conversion operator. It is by that proven that all investors have equal opportunity chances on the foreign exchanges independently of their home currency.

I emphasise that the necessity to apply the logarithmic norm on currencies has no bearing on their price behaviour. It gives no basis to conclude on the form of the probability distribution. Even there would, technically speaking, be a way to avoid switching to the logarithmic norm while guaranteeing overall consistency in returns between the currency perspectives. It was suggested by Chu (2005) as an intellectual thought. His suggestion is to calculate all returns from one currency perspective only and let all market participants adapt to that standard. It does solve the Siegel paradox, though it may be challenging to try to impose it on the world's investment community.

The Siegel paradox is an old problem in fact, much older than the author and of much broader concern. Fisher (1922) has devoted an entire book on the consistency problems encountered when measuring and comparing returns. For the anecdote, Walsh (1921) describes a heated debate that has taken place in seventeenth-century Florence on in essence the same issue: *if a horse worth 100 crowns is valued by someone at 1000 crowns an by someone else at 10 crowns, would their valuation error be of the same magnitude?* No easy answer has ever been given.

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