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Modelling Tail Risk in a Continuous Space

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About the author

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Didier MAILLARD is Senior Advisor to Amundi on Research. He is since 2001 Professor at Conservatoire national des arts et metiers (CNAM), where he holds a Chair of Banking. Previously, he has been an economist at the French Ministry of Finance and at the OECD (1980-1992) – economic forecasts, economic policy, public finance, tax studies, financial sector - and has occupied various positions at Paribas (and then BNP Paribas) from 1992 to 2001: chief economist, head of asset management, risk advisor. He is a graduate from Ecole polytechnique (Paris) and Ecole nationale d'administration.

His main fields are portfolio optimization, asset management, wealth management and tax incidence (in particular on investment return).

Abstract

Non normal distributions are a fact of life. In the financial world, many distributions display tail risk, i.e. (negative) skewness and excess kurtosis. Being able to model such risk is useful in various and important fields: risk measurement, fund management performance evaluation, asset pricing…

One way to model tail risk is to introduce discontinuities, such as jumps, to describe the distribution of values or returns. It is however possible, and often convenient, to model tail risk in a continuous space.

Both Cornish-Fisher and Gramm-Charlier expansions (which is the simple form of a family of Edgeworth expansions) are means to transforming a Gaussian distribution into a non-Gaussian distribution, the skewness and the kurtosis of which can be controlled if the transformations are properly implemented. This may be useful for modelling distributions for a wide range of issues, especially in risk assessment and asset pricing.

The expansions differ in their nature: Cornish-Fisher is a transformation of a random variable, or of quantiles, meanwhile Gramm-Charlier is a transformation of a probability density. Both transformations must be implemented with care, as their domain of validity does not cover the whole range of possible skewnesses and kurtosis. It appears that the domain of validity of Cornish-Fisher is much wider that the domain of validity of Gramm-Charlier.

This, and the fact that Cornish-Fisher provides easily the quantiles of the distribution, gives it an advantage over Gramm-Charlier in several configurations.

Keywords: Risk, variance, volatility, skewness, kurtosis, non-Gaussian distribution .

JEL classification: C02, C51, G11, G32

1 – Introduction

The Cornish Fisher expansion is a way of transforming a Gaussian distribution into a non-Gaussian distribution, the skewness and the kurtosis of which can be controlled if the transformation is properly implemented (Maillard, 2012). The Gramm Charlier expansion is also a way of transforming a Gaussian distribution into a non-Gaussian one, with the desired skewness and kurtosis.

Those expansions may prove very useful to model uncertain variables or events which obviously are not normally distributed. In the field of finance, one observes that return or changes in asset prices distributions display (generally negative) skewness and (generally positive) excess kurtosis. These moments should be taken into account when risk is assessed, and in asset pricing.

Gramm-Charlier has been used in option pricing, for instance in a seminal paper by Corrado & Sue (1996). Cornish-Fisher has been used in several papers considering the risk of asset returns: see for example Cao & alii (2010), or Fabozzi & alii (2012). Maillard (2013b) uses Cornish-Fisher to estimate the cost of tail risk in a managed portfolio according to a manipulation-proof performance measure. Aboura & Maillard (2014) use Cornish-Fisher for the purpose of option pricing.

This paper attempts to assess the respective merits of both transformations.

2 – Nature of the expansions

Cornish-Fisher and Gramm-Charlier expansions are not about the same object. The Cornish-Fisher expansion is a transformation of a standard Gaussian random variable *z* into a non-Gaussian variable *Z*, such that:

$$(CF) \qquad Z = z + (z^2 - 1)\frac{S}{6} + (z^3 - 3z)\frac{K}{24} - (2z^3 - 5z)\frac{S^2}{36}$$

Where *S* and *K* are parameters tied to skewness and kurtosis respectively.

The Gramm-Charlier expansion transforms a standard Gaussian probability density φ into a non-Gaussian probability density Φ, such that:

$$(GC) \quad \Phi(z) = \varphi(z)\left[1 + \frac{S}{6}H_3(z) + \frac{K}{24}H_4(z)\right] = \varphi(z)\left[1 + \frac{S}{6}(z^3 - 3z) + \frac{K}{24}(z^4 - 6z^2 + 3)\right]$$

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

H_n is the Hermite polynomial of order n

$$H_n(z) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$$

Note that the term of « expansion » is related to the fact that the probability laws derived from the Cornish-Fisher or Gramm-Charlier expansions are « approximations » of any probability law displaying the same four moments. But they are not in themselves approximate probability laws, provided they lie within a certain domain of validity.

3 – Moments

Moments are easy to compute in the case of Gramm-Charlier (see for example Jondeau and Rockinger (2001), or Appendix).

It ensues that volatility is unitary and that skewness and excess kurtosis[1] are equal to the S and K parameters respectively:

$$(GC)$$
$$\sigma = 1$$
$$\hat{S} = S$$
$$\hat{K} = K$$

[1] We mean by excess kurtosis the difference between kurtosis and 3, which is the kurtosis of a Gaussian distribution.

The computation of moments for Cornish-Fisher is rather more complex, and has been done by Maillard (2012).

$$
\begin{aligned}
&(CF)\\
&\sigma = \sqrt{1+\frac{1}{96}K^2+\frac{25}{1296}S^4-\frac{1}{36}KS^2}\\
&\hat{S} = \frac{S-\frac{76}{216}S^3+\frac{85}{1296}S^5+\frac{1}{4}KS-\frac{13}{144}KS^3+\frac{1}{32}K^2S}{\left(1+\frac{1}{96}K^2+\frac{25}{1296}S^4-\frac{1}{36}KS^2\right)^{1.5}}\\
&\hat{K} == \frac{\left[\begin{array}{l} 3+K+\frac{7}{16}K^2+\frac{3}{32}K^3+\frac{31}{3072}K^4-\frac{7}{216}S^4-\frac{25}{486}S^6+\frac{21665}{559872}S^8-\frac{7}{12}KS^2 \\ +\frac{113}{452}KS^4-\frac{5155}{46656}KS^6-\frac{7}{24}K^2S^2+\frac{2455}{20736}K^2S^4-\frac{65}{1152}K^3S^2 \end{array}\right]}{\left(1+\frac{1}{96}K^2+\frac{25}{1296}S^4-\frac{1}{36}KS^2\right)^2}-3
\end{aligned}
$$

Except for very small (absolute) values for S and K, volatility differs (slightly) from 1, and skewness and kurtosis differ (sometimes hugely) from the skewness and kurtosis parameters.

It is thus possible to build distributions with the desired four first moments with both expansions. Choosing the parameters is straightforward in the Gramm-Charlier case. It is somewhat more arduous in the Cornish-Fisher case: one has to compute the S and K parameters by reversing the two expressions giving the actual skewness and kurtosis (which must be done numerically), and then correct the Cornish-Fisher expansion by dividing by the value of volatility, to obtain a random variable with unitary variance and the desired skewness and volatility.

4 – Probability densities

By definition, the probability density corresponding to Gramm-Charlier is given by the expansion:

$$(GC) \quad \Phi(z) = \varphi(z)\left[1 + \frac{S}{6}H_3(z) + \frac{K}{24}H_4(z)\right] = \varphi(z)\left[1 + \frac{S}{6}(z^3 - 3z) + \frac{K}{24}(z^4 - 6z^2 + 3)\right]$$

The probability density corresponding to Cornish-Fisher may be computed from the definition of the random variable (see Maillard (2013a)), but its expression is somewhat more complex:

$$(CF) \quad \Phi(Z) = \frac{\varphi(z)}{z^2\left(\frac{K}{8} - \frac{S^2}{6}\right) + z\frac{S}{3} + 1 - \frac{K}{8} + \frac{5S^2}{36}}$$

with:

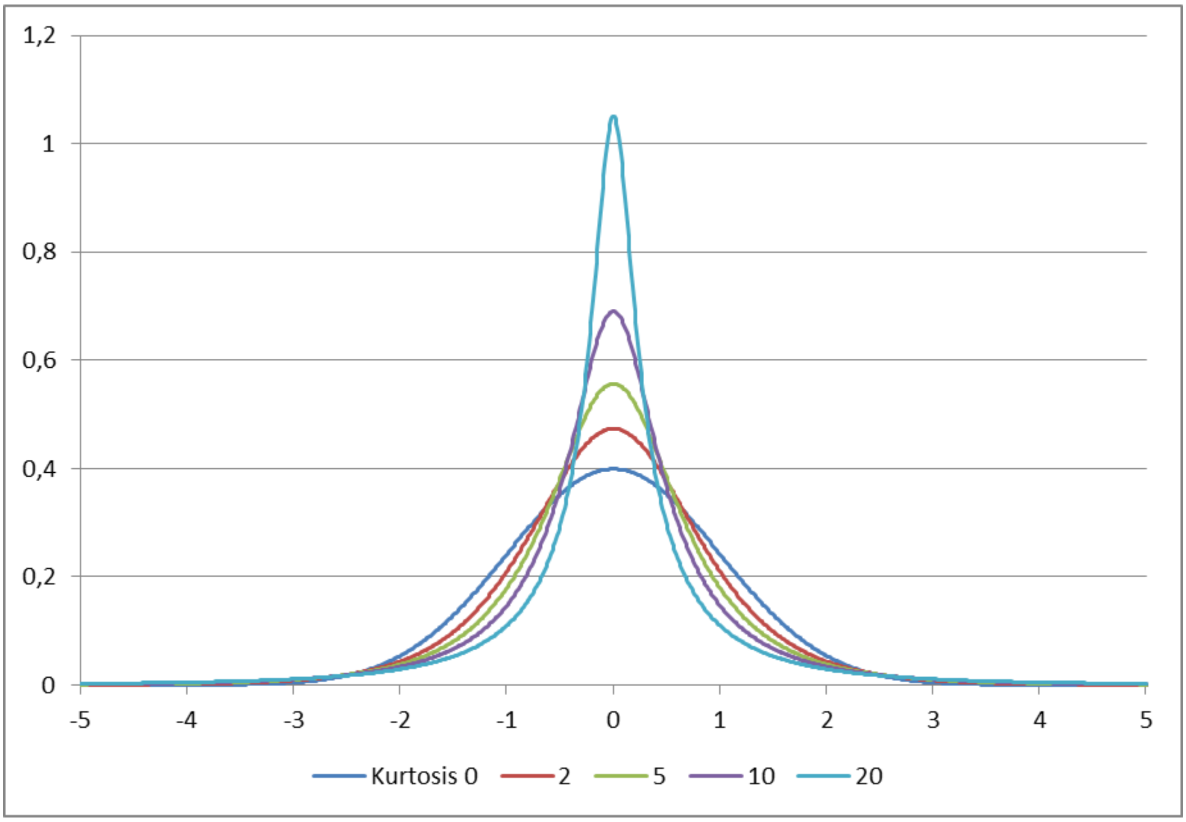
$$z = \zeta_{SK}(Z) = a'_0/3 + \sqrt[3]{\frac{-q' + Z/a_3 + \sqrt{(q' - Z/a_3)^2 + \frac{4}{27}p^3}}{2}} + \sqrt[3]{\frac{-q' + Z/a_3 - \sqrt{(q' - Z/a_3)^2 \frac{4}{27}p^3}}{2}}$$

$$a_3 = k - 2s^2 \quad a'_0 = \frac{-s}{k - 2s^2} \qquad k = \frac{K}{24} \qquad s = \frac{S}{6}$$

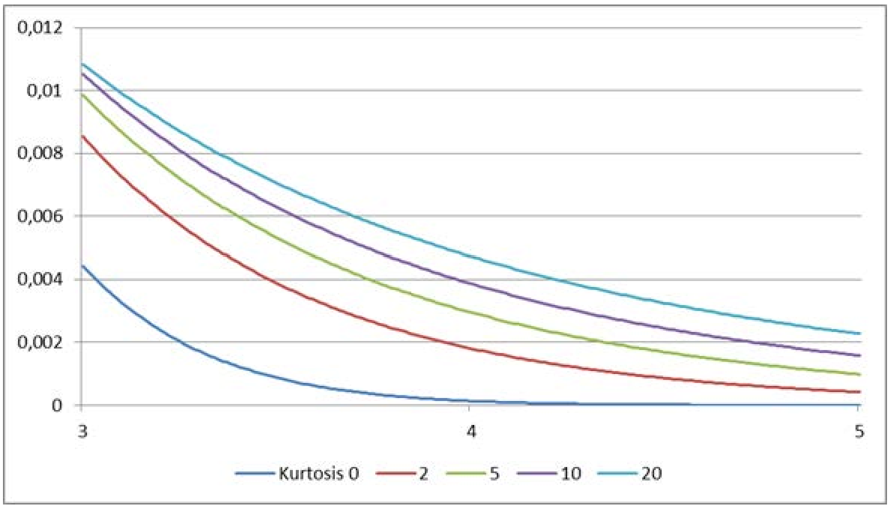
$$p = \frac{1 - 3k + 5s^2}{k - 2s^2} - \frac{1}{3}\frac{s^2}{\left(k - 2s^2\right)^2} \qquad q' = \frac{-s}{k - 2s^2} - \frac{1}{3}\frac{s(1 - 3k + 5s^2)}{\left(k - 2s^2\right)^2} - \frac{2}{27}\frac{s^3}{\left(k - 2s^2\right)^3}$$

Though complex, this formula may be encapsulated into a single spreadsheet cell.

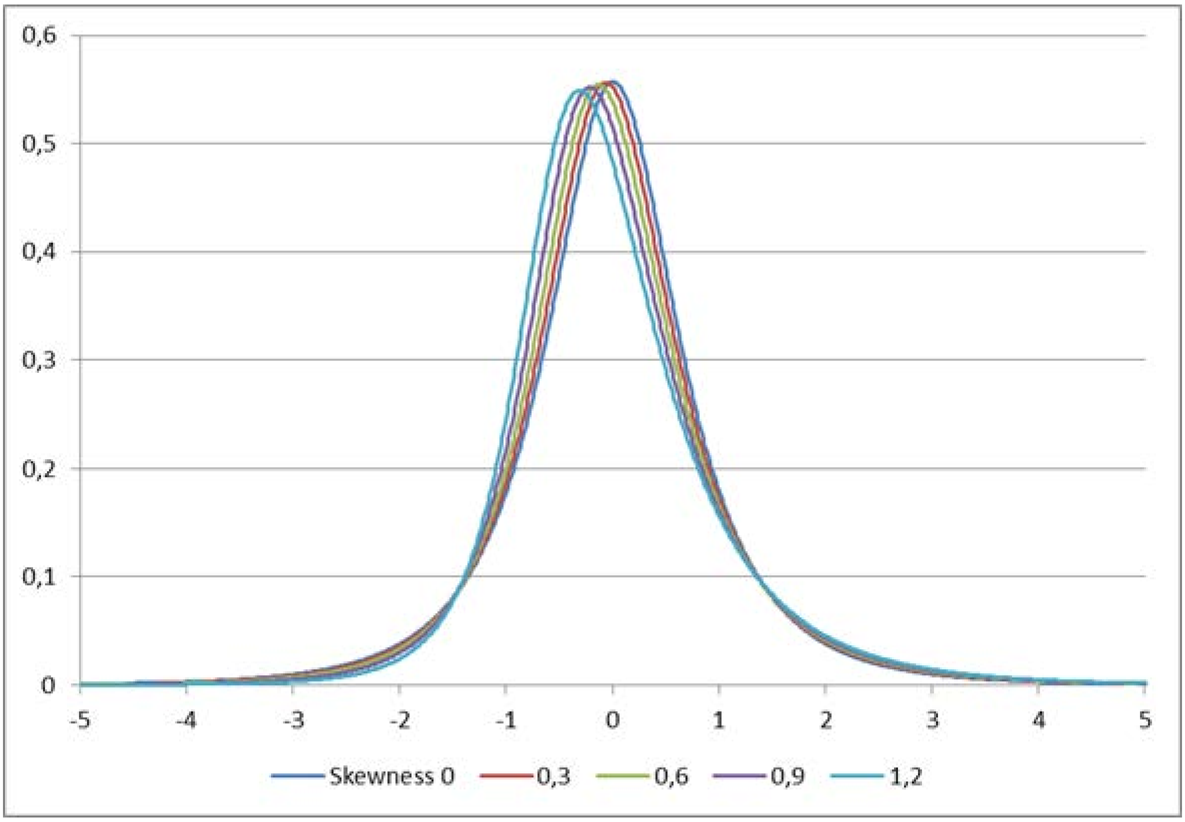
It is possible to plot the density function and its deformation according to kurtosis (see Maillard (2013a) for more details), for the whole distribution.



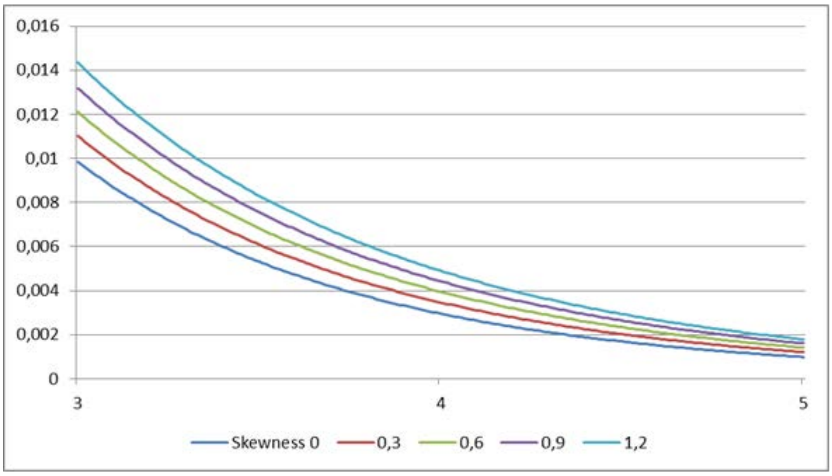
and for the tail of the distribution (here the right-hand tail),



The impact of skewness (here positive skewness) is less visible on the whole distribution,



It appears however clearly on the tail (here the right hand tail):



Those probability densities are useful for computing numerically integrals, for instance in issues of option pricing, and this seems to confer Gramm-Charlier an advantage of simplicity. However, such integrals may also be computed using the quantiles of the distribution (by equipondering their values). In that case, the advantage goes to Cornish-Fisher, which gives an immediate value of the quantiles. In Gramm-Charlier case, there is no simple expression of the quantiles.

5 – Domain of validity

This is a very important point. Any system of probability should present two features:

- Non-negativity: any possible event should have a probability equal or superior to zero,
- Unitary sum: the probabilities of all possible events should add to one.

For the Gramm-Charlier expansion, which is expressed in terms of a probability density, it means that:

$$(GC)\quad \Phi(z)=\varphi(z)\left[1+\frac{S}{6}H_3(z)+\frac{K}{24}H_4(z)\right]=\varphi(z)\left[1+\frac{S}{6}(z^3-3z)+\frac{K}{24}(z^4-6z^2+3)\right]\geq 0\quad \forall z$$

$$\int_{-\infty}^{+\infty}\Phi(z)dz=1$$

It is easy to show (see Appendix, moment M_0) that the second condition is fulfilled. As for the first one, a 4th-order polynomial has to be always positive. The condition therefore has been studied in particular by Jondeau & Rockinger (2001).

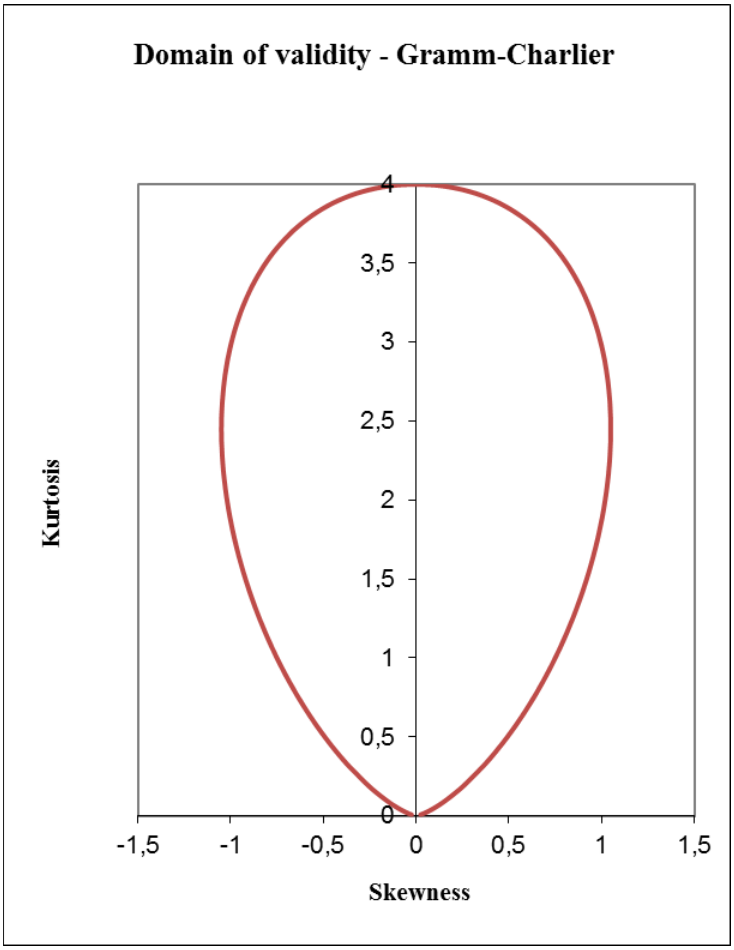
They prove that the boundary of the domain of validity has a parametric definition given by:

$$\hat{S}(z)=-24\frac{H_3(z)}{z^6-3z^4+9z^2+9}$$

$$\hat{K}(z)=72\frac{H_2(z)}{z^6-3z^4+9z^2+9}$$

It is possible to plot this boundary in the skewness/kurtosis plane. The domain of validity is the inner part of the boundary.

Chart 1



It appears that kurtosis cannot exceed 4, and skewness 1.05 in absolute value. The domain of validity is thus quite limited. It is not rare to observe in returns distribution kurtosis in excess of 4 and skewnesses in excess of 1 in absolute terms[2].

For Cornish-Fisher, the probability density adds to 1 by definition. The non-negativity condition is equivalent to the monotonicity of the transformation of the quantiles, i.e. that (the positive sign resulting from the fact that Z is positive for large positive values of z):

[2] Gramm-Charlier is the simplest (lowest polynomial order) of a family of density expansions known as the Edgeworth expansions. The higher order expansions are also subject to nonnegativity problems. In addition, their sum does not necessarily equal 1.

$$\frac{dZ}{dz} > 0 \quad \forall z$$

The condition for that to hold is that *S* and *K* are subject to the following inequality:

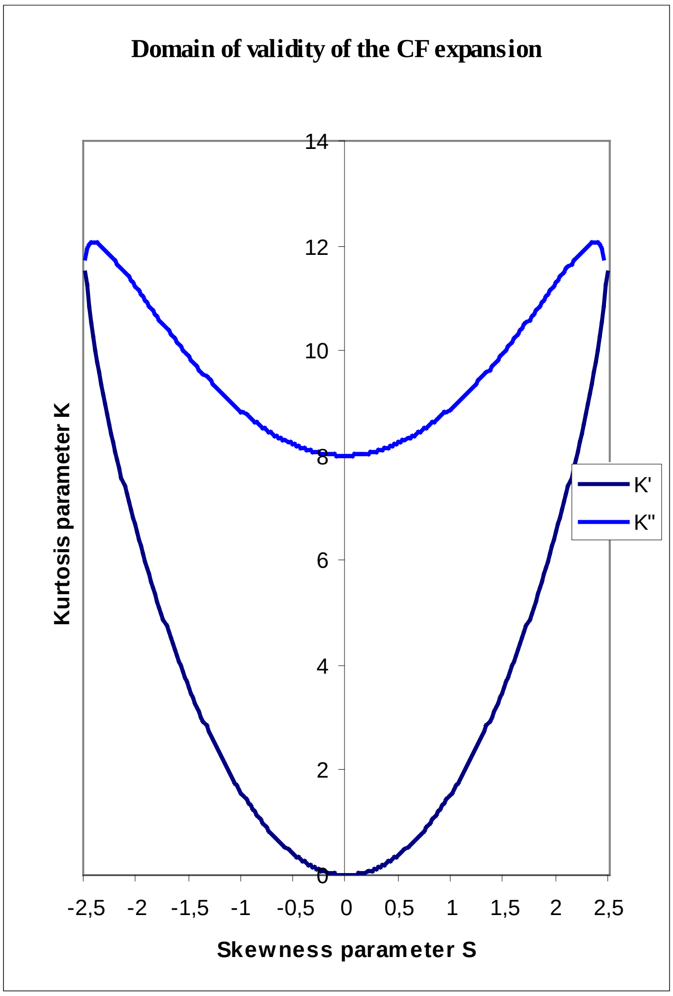
$$\frac{S^2}{9} - 4\left(\frac{K}{8} - \frac{S^2}{6}\right)\left(1 - \frac{K}{8} + \frac{5S^2}{36}\right) \leq 0$$

This result is known for a long time. It is interesting to rewrite it (see Maillard (2012), with a presentation of the derivation) as:

$$9k^2 - (3 + 33s^2)k + 30s^4 + 7s^2 \leq 0 \quad k = \frac{K}{24} \quad s = \frac{S}{6}$$
$$\frac{1 + 11s^2 - \sqrt{s^4 - 6s^2 + 1}}{6} \leq k \leq \frac{1 + 11s^2 + \sqrt{s^4 - 6s^2 + 1}}{6}$$

It implies that S cannot exceed 2.485 (for the square root to be real), and it gives an equation of the boundary in the (*S*,*K*) plane.

Chart 2

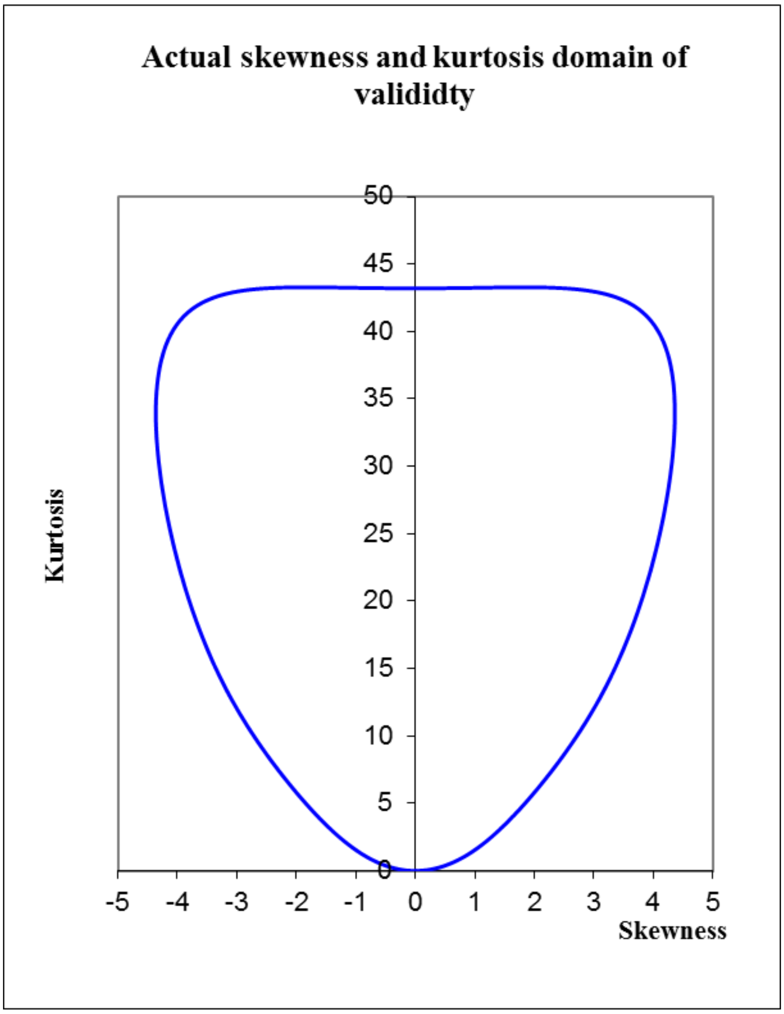


However, one should remember at this stage that the skewness and kurtosis parameters are not the actual skewness and kurtosis. The equation of the boundary in the ($\hat{S}$, $\hat{K}$) plane could be obtained by reversing the relationship but it is not easily tractable.

However, one can obtain a parametric representation of the boundary using *S* as a parameter. S leads to $\hat{S}$, and to $\hat{K}$ through *K*.

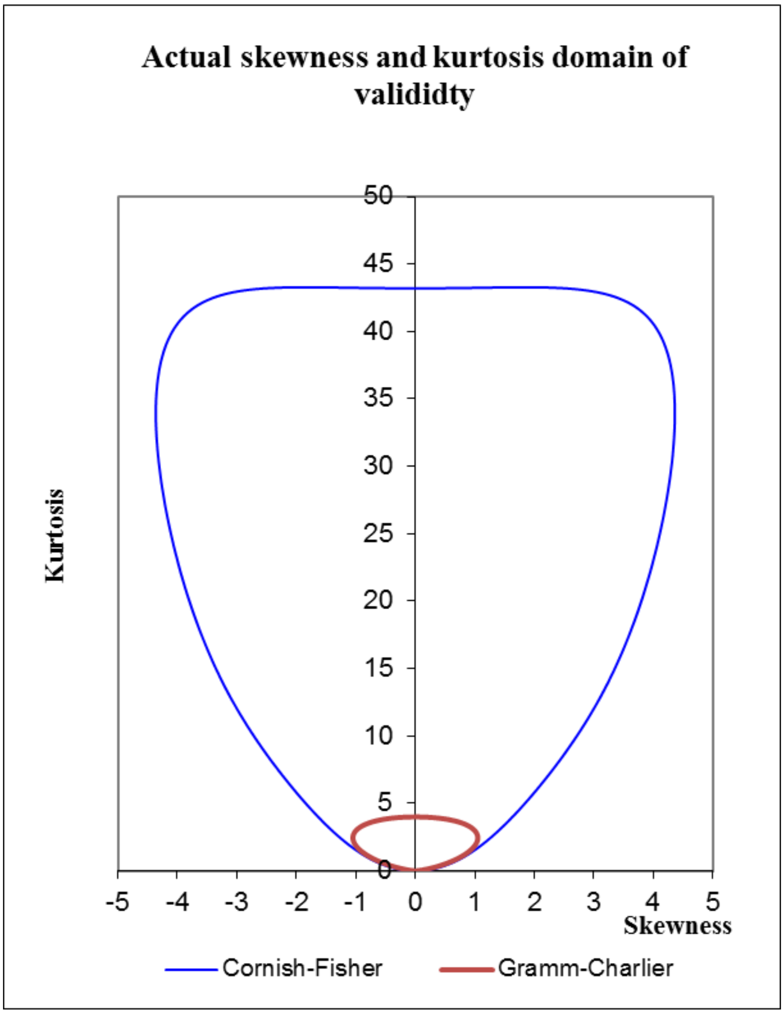
This gives in Chart 3 the boundary of the domain of validity of the Cornish-Fisher transformation.

Chart 3



The domain of validity is much wider in the Cornish-Fisher case, as may be seen in chart 4.

Chart 4



There have been proposals to extend the domain of validity of the transformation by "rectifying" them: see in the case of Gramm-Charlier Jondeau & Rockinger (2001), or more generally Chernozhukov & alii (2007). Their aim is to correct the breaches of non-negativity. Those rectifications lead to new distribution laws, which may not be as parsimonious in their implementation as Cornish-Fisher or Gramm-Charlier.

6 – Links with VaR and CvaR

Contrarily to Gramm-Charlier, Cornish-Fisher provides a simple expression of the quantiles of the distribution. It is therefore convenient to compute easily values at risk (VaR), which are tied to a quantile in the unfavourable part of the distribution (VaR is volatility times minus the quantile times expected value minus present value). Omitting those constants, VaR at threshold 1-a may be written as:

$$VaR_{1-\alpha} = -Z_{\alpha}$$

It may be shown easily (see Maillard (2012)), using a Cornish-Fisher expansion, that:

$$VaR_{1-\alpha} = v_{\alpha} + (1 - v_{\alpha}^{2})\frac{S}{6} + (5v_{\alpha} - 2v_{\alpha}^{3})\frac{S^{2}}{36} + (v_{\alpha}^{3} - 3v_{\alpha})\frac{K}{24}$$

where $v_{\alpha} = -z_{\alpha} = N^{-1}(\alpha)$ is value-at-risk in the Gaussian case.

It is also easy to obtain another, more consistent, measure of risk, the conditional value at risk (CVaR)

$$CVaR_{1-\alpha} = y_{\alpha}\left[1 - v_{\alpha}\frac{S}{6} + (1 - 2v_{\alpha}^{2})\frac{S^{2}}{36} + (-1 + v_{\alpha}^{2})\frac{K}{24}\right]$$

where $y_{\alpha} = \frac{1}{\alpha}\frac{1}{\sqrt{2\pi}}e^{-\frac{z_{\alpha}^{2}}{2}}$ is the conditional value-at-risk in the Gaussian case.

Cornish-Fisher provides thus a simple method for correcting risk measures that would prevail in a Gaussian situation for skewness and kurtosis.

A caveat: The expressions of VaR and CVaR depend on the skewness and kurtosis **parameters**. Those should be obtained by reversing the two expressions giving the actual skewness and kurtosis as a function of the skewness and kurtosis parameters.

7 – Conclusions

Due to its much wider domain of validity, Cornish-Fisher should be preferred in most cases. It has also the advantage of giving a simple expression of the quantiles, which may be quite useful in numerical simulations.

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Appendix

Gramm-Charlier moments

$$M_i = \int_{-\infty}^{+\infty} z^i \Phi(z) dz = \int_{-\infty}^{+\infty} \left[z^i + \frac{S}{6}(z^{i+3} - 3z^{i+1}) + \frac{K}{24}(z^{i+4} - 6z^{i+2} + 3z^i) \right] \varphi(z) dz$$

$$M_i = \int_{-\infty}^{+\infty} \left[z^i (1 + K/8) + z^{i+1}(-S/2) + z^{i+2}(-K/4) + z^{i+3}(S/6) + z^{i+4}(K/24) \right] \varphi(z) dz$$

$$M_i = (1 + K/8) m_i - (S/2) m_{i+1} - (K/4) m_{i+2} + (S/6) m_{i+3} + (K/24) m_{i+4}$$

Where m_i is the i-th order moment of a standard Gaussian distribution[3]

$$M_0 = (1+K/8)m_0 - (S/2)m_1 - (K/4)m_2 + (S/6)m_3 + (K/24)m_4 = 1 + K/8 - K/4 + 3K/24 = 1$$
$$M_1 = (1+K/8)m_1 - (S/2)m_2 - (K/4)m_3 + (S/6)m_4 + (K/24)m_5 = -S/2 + 3S/6 = 0$$
$$M_2 = (1+K/8)m_2 - (S/2)m_3 - (K/4)m_4 + (S/6)m_5 + (K/24)m_6 = 1 + K/8 - 3K/4 + 15K/24 = 1$$
$$M_3 = (1+K/8)m_3 - (S/2)m_4 - (K/4)m_5 + (S/6)m_6 + (K/24)m_7 = -3S/2 + 15S/6 = S$$
$$M_4 = (1+K/8)m_4 - (S/2)m_5 - (K/4)m_6 + (S/6)m_7 + (K/24)m_8 = 3 + 3K/8 - 15K/4 + 105K/24$$
$$M_4 = 3 + K$$

It follows that volatility is unitary and that skewness and kurtosis are equal to the S and K parameters respectively:

$$\sigma = 1$$
$$\hat{S} = S$$
$$\hat{K} = K$$

[3] $m_1 = m_3 = m_5 = m_7 = 0 \quad m_0 = m_2 = 1 \quad m_4 = 3 \quad m_6 = 15 \quad m_8 = 105$

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