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Optimal Allocation in the S&P 600 under Size-Driven Illiquidity

Research & Macro Strategy

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Abstract

A number of empirical studies have investigated how mutual funds do react to incoming financial resources. As long as liquidity constraints are narrow, fund managers tend to upscale already existing positions without looking for new investment opportunities.

Concomitantly, performance of funds decreases which is explained by the expansion of size-driven liquidity costs. We put forward a model of asset allocation that accounts for market liquidity frictions and recovers the inverse relation between fund size and fund performance. The model prescribes how fund portfolio managers should react as financial resources enter the fund. We estimate the model on S&P 600 stock data and investigate optimal allocation behavior under fund size increase. We obtain that to confine the negative effect of liquidity frictions on performance, portfolio diversification should be increased under fund size increase. In particular, once a certain fund size is attained, fund managers should incorporate new investments which enhances portfolio diversity and reduces liquidity-driven performance erosion.

Keywords: Allocation, diversification, fund performance, fund size, market liquidity

JEL classification: G11, G12, G20, G23

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1 Introduction

Mutual funds may increase their size through incoming financial resources provided by investors. Correspondingly, fund portfolio managers need make a decision about the way they invest new cash holdings. A first option is to simply upscale all positions already held keeping weights unchanged. Alternatively, they may alter portfolio proportions keeping investing in the same assets. A third possibility they have is to increase the set of investments which necessarily involves portfolio weights rebalancing. These three alternatives all entail different results on funds performance, notably due to the existence of liquidity frictions.

Empirical literature has extensively dealt with the effects of fund size increase on performance in the active money management industry. Grinblatt and Titman (1989) find some evidence that gross fund returns decline with fund size. Indro *et al.* (1999) established that fund performance behaves in a concave manner with fund size as it first increases and then diminishes. Considering various performance benchmarks, Chen *et al.* (2004) document that fund returns decline with lagged fund size. Using an alternative performance measurement methodology than that of Chen *et al.* (2004), the paper by Yan (2008) corroborates a significant inverse relation between fund size and fund performance.

Most investigations explain the adverse relation between fund size and fund performance by the presence of market liquidity issues. Perold and Salomon (1991) contend that large transactions in active fund management cause important price impacts leading to diseconomies of scale. Chen *et al.* (2004) find that the inverse relation between size and performance is stronger among small-cap funds. This supports the hypothesis that fund size erodes performance to the extent that small-cap stocks are less liquid. In addition, Yan (2008) documents that the inverse relation is more pronounced for funds that hold less liquid portfolios.

However, liquidity by itself is not sufficient to explain that fund size affects performance negatively. Chen *et al.* (2004) show that in addition to liquidity issues, organizational diseconomies related to hierarchy costs also play an important role in performance reduction. Further, Chan and Lakonishok (1995) and Keim and Madhavan (1997) document that funds with high demands for immediacy are associated with larger market impact and greater trading costs. Relying on these results, Yan (2008) provides the evidence that the negative relation between fund size and fund performance is more pronounced among growth and high turnover funds that tend to have high demands for immediacy.

Pollet and Wilson (2008) examine how size affects mutual fund behavior rather than directly performance. They document that growing funds primarily increase ownership shares in companies they already own. This suggests that portfolio managers seek new investment opportunities only when liquidity constraints become siginificant. This behavior is presented as the proximate cause of diminishing returns to scale for mutual funds. An other finding in Pollet and Wilson (2008) is that the beneficial effect of diversification on performance is more pronounced for funds that invest in the small-cap sector. These results support liquidity constraints as an explanation for why large-cap funds diversify more slowly in response to growth in assets under management (Pollet and Wilson, 2008).

The exposed literature thus clarifies *how* funds *do react* to size upgrading and assesses the impact on their performance. Little attention seems to have been paid on *how* funds *should react* to size-driven liquidity frictions. This problem entails modelling the logical linkage connecting investment size, portfolio composition, and liquidity effects on value and performance. We develop in this paper a model which enables incorporating market liquidity frictions in performance assessment. This model recovers the stylized fact that liquidity frictions entail a negative relation between size and performance. Its main contribution is to prescribe how funds' portfolio managers *should* invest incoming funds among alternative investment opportunities in the presence of liquidity issues.

We propose a theoretical setup that enables comparing mean-variance maximizing allocations in two alternative settings. A first setting, that we refer to as *marked-to-market* is insensitive to market liquidity frictions and its output is the usual mean-variance maximizing portfolio choice. An other setting, which is assigned the attribute *liquidity-adjusted*, incorporates market liquidity frictions in optimal portfolio choice. In our modelling market liquidity frictions are driven by fund size, which entails that there is an optimal allocation specific to each fund size. Two items generate liquidity frictions in the model. First, it relies on the critical concept of *risk rule* which requires fund's positions to fulfill some specific risk constraints. The requirement for the fund to satisfy the risk rule may lead to buying or selling units of selected assets which entails an update of invested weights. Trading activity then appeals the second item called *supply-demand curve*, which for each equity assigns a transaction price to any quantity that is traded. The model is tested through a number of experiments undertaken in a controlled environment designed for analyzing comparative statics among varying assumptions of the underlying parametric setting.

As a next step we estimate the model on a large dataset of stock trades. These stocks are the constituents of the S&P 600 index. The model is then used to conduct a number of calibrated experiments. In particular, we investigate optimal allocation behavior under fund size increase for varying widths of the set of investment opportunities. Comparative statics and calibrated experiments yield the following main outcomes. First, in the liquidity-adjusted setting portfolio diversification within stocks is enhanced compared to the setting where market liquidity is not considered and increases concomitantly with fund size. This diversification increase means that liquidity-adjusted allocation alters weights as fund size rises to restrict the negative effect of liquidity frictions on *ex-ante* performance. Next, given a certain fund size augmentation, diversification enhancement within stocks may occur in two different manners. First, it can be achieved by a revision of weights in already existing investments. Alternatively, weights may be altered by new investments in assets belonging to the considered set of investment opportunities. We find that this latter behavior usually occurs for large funds which is in line with the empirical finding by Pollet and Wilson (2008) that managers seek new investment opportunities only when liquidity constraints become significant. Finally, proportion of wealth invested in the risk-free asset augments with fund size. This behavior becomes particularly significant for large funds that invest in a restricted set of investment opportunities.

We previously exposed the main contribution of this paper to extant literature which is that it prescribes how fund portfolio managers *should* invest incoming capital among alternative investment opportunities in the presence of liquidity issues. It also contributes to two others streams of existing literature.

First, it enriches the modelling of liquidity problems by linking market liquidity to funds'

internal rules or regulatory constraints applying to portfolios. This recalls Brunnermeier and Pedersen (2008) who put forward a model which relates an asset's market liquidity to traders' funding liquidity. In their paper, funding liquidity is affected by capital margins on risky securities. They notably show that under certain conditions both kinds of liquidities are mutually reinforcing which leads to liquidity spirals. Our approach differs in the fact that assets' market liquidity is exogeneous and may be assessed on market data. Further, liquidity frictions are driven by the requirement for fund's positions to fulfill some internal rules or regulatory risk constraints which involves trading.
Next, it expands literature dealing with portfolio management under liquidity issues. Brown *et al.* (2010) analyze the problem of an investor who needs to unwind a portfolio in the face of recurring and uncertain liquidity needs with a model that accounts for the price impact of trading. While they deal with optimal liquidation, the present paper contrariwise addresses the question of optimal allocation under fund size increase in the presence of liquidity frictions. An other contribution by Longstaff (2001) solves an investor's intertemporal portfolio choice problem under liquidity constraints where liquidity constraints are modelled as trading strategies of bounded variation. It is obtained that a liquidity-constrained investor acts as if facing borrowing and short-selling constraints, and may take riskier positions than in liquid markets. The current paper retains an alternative path as it models internal risk management rules which affect liquidity of the portfolio. Hence optimal allocation is adjusted to both market liquidity and risk rules.

The remainder of the paper is organized as follows. Section 2 develops the theoretical framework. In section 3 we test the economic relevance of the model by conducting a number of numerical experiments under a variety of parametrizations (comparative statics). Section 4 introduces the data and details data preparation required prior to model calibration. The methodology put forward to estimate model parameters is presented in section 5 and estimates are also provided. A number of empirical experiments are conducted in section 6. Finally, section 7 concludes the paper.

2 Theoretical setup

Consider the case of an investment fund who needs to allocate financial resources between the different assets composing a given set of investment opportunities. Letting this latter set be composed of one risk-free asset and N risky assets, a portfolio is then any vector $\boldsymbol{q} = (q^0, q^1, ..., q^N)$ where $q^i \in \mathbb{R}$ denotes the quantity of asset $i \in \{0, 1, ..., N\}$ held. Since asset $i = 0$ is the risk-free asset, q^0 denotes an amount of cash holdings. To allocate its wealth, the fund maximizes a mean-variance performance metric over a one-period time interval. Time 0 denotes present time, that is the time at which the allocation is performed. Time 1 is the last day of the management period, that is the time horizon under which the allocation of wealth is conducted. The duration between time 0 and time 1 is given by Δ. For any time $t \in \{0, 1\}$, the mid-price of equity i is denoted by S_t^i. At time 0, the time 1 mid-price of equity i is random and given by

$$S_1^i = S_0^i \exp\left\{(\mu^i - (\sigma^i)^2/2)\Delta + \sigma^i \sqrt{\Delta}\epsilon^i\right\} \tag{1}$$

where $\mu^i \in \mathbb{R}$ and $\sigma^i > 0$ denote respectively drift and volatility of the mid-price and where ϵ^i is a standard normal random variable. The risk-free asset delivers an annual interest rate r_f which is continuously compounded. Letting $E[.]$ and $V[.]$ denote respectively expectation and variance and R any random return, mean-variance is defined by $\text{MV}[R] := E[R] - \frac{\theta}{2}V[R]$ where $\theta > 0$ denotes investors' risk aversion. An overview of the subsequent model presentation is pictorially depicted by figure 8 in appendix A.

2.1 Marked-to-market optimal allocation

Let us first derive the optimal allocation problem when market liquidity frictions are not taken into consideration by the fund's portfolio manager. This setting, referred to as *marked-to-market* setting, is used as benchmark in the present study. It makes sense to consider marked-to-market valuation when a portfolio is not currently traded as it is then not exposed to market liquidity frictions. For any portfolio $\boldsymbol{q}$, the time 0 *marked-to-market value* is given by a function $V_0^M : \mathbb{R}^{N+1} \to \mathbb{R}$ defined by $V_0^M(\boldsymbol{q}) := q^0 + \sum_{i=1}^N q^i S_0^i$ with S_0^i being the mid-price of equity $i \in [1, N]$. Remark that V_0^M increases linearly on $\mathbb{R}^{N+1}$ which reflects the absence of liquidity frictions in the marked-to-market setting. Given the risk-free yield and the variation of mid-prices over period Δ, for any portfolio $\boldsymbol{q}$ chosen at time 0 the time 1 marked-to-market value is given by $V_1^M(\boldsymbol{q}) = q^0 \exp(r_f \Delta) + \sum_{i=1}^N S_1^i q^i$.

To allocate fund's wealth at time 0, the portfolio manager maximizes mean-variance of portfolio marked-to-market return where the latter is defined by $R_\Delta^M(\boldsymbol{q}) := (V_1^M(\boldsymbol{q}) - V_0^M(\boldsymbol{q}))/V_0^M(\boldsymbol{q})$. Then, letting $C(.,.)$ denote date 0 covariance between any two random variables, portfolio mean-variance is given by $\mathrm{MV}[R_\Delta^M(\boldsymbol{q})] = E[R_\Delta^M(\boldsymbol{q})] - \frac{\theta}{2} V[R_\Delta^M(\boldsymbol{q})]$ with

$$E[R_\Delta^M(\boldsymbol{q})] = \frac{q^0(\exp(r_f\Delta) - 1) + \sum_{i=1}^N q^i \left(E[S_1^i] - S_0^i\right)}{q^0 + \sum_{i=1}^N S_0^i q^i}$$

and

$$V_0[R_\Delta^M(\boldsymbol{q})] = \frac{1}{\left(q^0 + \sum_{i=1}^N S_0^i q^i\right)^2} \times \left(\sum_{i=1}^N (q^i)^2 V_0(S_1^i) + 2 \sum_{i \neq j}^N q^i q^j C(S_1^i, S_1^j) \right).$$

Finally, the optimal portfolio choice problem in the marked-to-market setting consists in determining within the set of investment opportunities the asset holdings that maximize *ex-ante* performance. Assume that at time 0 fund wealth that must be allocated between the different investment opportunities is given by W_0. Then, the optimal allocation is the portfolio $\boldsymbol{q}^M$ which is the maximizing argument of the problem:

$$\underset{\boldsymbol{q} \in \mathbb{R}^{N+1}}{\mathrm{Max}} \mathrm{MV}\left[R_\Delta^M(\boldsymbol{q})\right] \text{ subject to } V_0^M(\boldsymbol{q}) = W_0,\ q^i \geq 0,\ i \in [0, N]. \tag{2}$$

Notice that the optimal allocation problem only allows for long positions.

2.2 Liquidity-adjusted optimal allocation

Let us now consider the problem of optimal wealth allocation in a setting that incorporates market liquidity frictions, that we refer to as *liquidity-adjusted* setting. In this setting the portfolio manager still allocates wealth by maximizing *ex-ante* mean-variance of portfolio return between dates 0 and 1. However, for risk management purposes, he acts in a forward-looking way and carries out an adjustment to time 1 value given information available at time 0. Several items must be introduced to derive time 1 liquidity-adjusted value that we detail in the following paragraphs.

A first critical notion of the model is the concept of *supply-demand curve*. The supply-demand curve assigns to any traded quantity the transaction price that is obtained for this quantity. Let S denote the mid-price, that is the price quoted when there is no trade. Traded quantity is indicated by $x \in \mathbb{R}$ where we use the convention that $x > 0$ is a sell order and $x < 0$ a buy order. Then, an asset's *supply-demand curve* is a function $\mathcal{S}(.)$ defined by $\mathcal{S}(x) := S - \alpha x$ where α is a strictly positive parameter describing the asset's liquidity. Note that when no trade occurs ($x = 0$) the price of the considered asset is given by the mid-price.

Contrarily to the marked-to-market setting, the *liquidation* setting assumes the sale of all positions held in the portfolio. Precisely, the *liquidation value* gives the proceeds (resp. expenses) stemming from portfolio liquidation (resp. purchase) and incorporates market liquidity frictions. For any portfolio $\boldsymbol{q}$, the time 1 liquidation value is given by a map $V_1^L : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ defined by $V_1^L(\boldsymbol{q}) := q^0 + \sum_{i=1}^N q^i \mathcal{S}_1^i(q^i)$ where $\mathcal{S}_1^i(.)$ denotes asset i's supply-demand curve at time 1. Replacing $\mathcal{S}_1^i(q^i)$ by its expression, liquidation value may be rewritten as $V_1^L(\boldsymbol{q}) = q^0 + \sum_{i=1}^N q^i S_1^i - \sum_{i=1}^N \alpha^i (q^i)^2$. This latter expression unveils the nonlinear shape of V_1^L on $\mathbb{R}^{N+1}$ especially due to the quadratic term $\sum_{i=1}^N \alpha^i (q^i)^2$ which represents the cost of liquidity.

Marking-to-market is the most optimistic approach as it ignores liquidity frictions and hence provides the highest value a portfolio may attain at a given point of time. By contrast, the liquidation value provides portfolio value in a worst-case scenario where all positions are sold thus exposing the portfolio to severe liquidity frictions. But the fair portfolio value actually ranges between the marked-to-market value and the liquidation value and depends to a large extent on constraints the portfolio is required to satisfy. We consider two different risk management constraints that we introduce below.

The first constraint that is considered is referred to as *size rule*. The size rule requires from the portfolio manager that portfolio positions remain beneath a certain quantity threshold settled by the fund's risk management team. This internal rule reflects aversion towards market liquidity frictions and avoids holding excessively large positions. Formally, the size rule is defined by $\mathcal{R}^S := \Big\{ \boldsymbol{q} \in \mathbb{R}^{N+1} : q_i \leq \tau_i, \ i \in [1, N] \Big\}$ where τ_i denotes a positive quantity threshold the position invested in asset i should not exceed.

The second constraint we put forward is called *Value-at-Risk* (VaR) *rule*. Recall that given some management horizon, VaR is defined as the largest loss a portfolio may undergo at a certain confidence level. Let $\gamma \in (0, 1)$ denote a fixed confidence level and X the random value discrepancy of a portfolio over a given time period. VaR is then defined by $\mathrm{VaR}_\gamma(X) := -\inf \{y \in \mathbb{R} : \mathbb{P}[X \leq y] \geq 1 - \gamma\}$ where $\mathbb{P}$ denotes the historical probability distribution of random variable X. Let δ denote the time period over which VaR is assessed and let X_δ be given by $X_\delta = S_{t+\delta} - S_t$ where t is any point of time. Then, under the previously made assumption that mid-price S is driven by a geometric brownian motion with drift μ and standard deviation σ, it is easily shown that $\mathrm{VaR}_\gamma(X_\delta) = S_t \Big(1 - \exp\{(\mu - 0.5\sigma^2)\delta + \sigma\sqrt{\delta}\Phi^{-1}(1 - \gamma)\Big)$ where $\Phi(.)$ is the cumulative probability distribution of a standard normal random variable.

Before determining liquidity-adjusted allocation, the fund's portfolio manager acts in a forward-looking way and wishes to derive the fair time 1 portfolio value under the VaR rule. The VaR rule, that we denote $\mathcal{R}^V$, requests that for each position that is held, present (time 0) expectation of time 1 VaR must remain below a certain risk limit. It entails that the VaR rule is formally defined by $\mathcal{R}^V := \Big\{ \boldsymbol{q} \in \mathbb{R}^{N+1} : E_0[\mathrm{VaR}_\gamma(X^i_\delta q^i)] \leq \rho, \ i \in [1, N] \Big\}$ where $\rho \geq 0$ denotes a fixed risk threshold. According to the VaR formula derived in the previous paragraph, the VaR rule may also be rewritten as

$$\mathcal{R}^V = \Big\{ \boldsymbol{q} \in \mathbb{R}^{N+1} : E_0[S^i_1] \times (1 - \exp\{(\mu - 0.5\sigma^2)\delta + \sigma\sqrt{\delta}\Phi^{-1}(1 - \gamma)) \leq \rho, \ i \in [1, N] \Big\}.$$

Let us now introduce the main concept of the model that we refer to as *liquidity-adjusted value*. The liquidity-adjusted value provides the fair value of a portfolio given that it has to fulfill either the size rule or the VaR rule in the presence of liquidity frictions. Precisely, it is obtained as the solution of a convex optimization problem incorporating the considered risk management rule on the one hand and assets' supply-demand curves on the other hand. Letting $\mathcal{R}$ denote either the size rule ($\mathcal{R} = \mathcal{R}^S$) or the VaR rule ($\mathcal{R} = \mathcal{R}^V$), for any portfolio $\boldsymbol{q} \in \mathbb{R}^{N+1}$ the liquidity-adjusted value at time 1 is given by a function $V_1^{\mathcal{R}} : \mathbb{R}^{N+1} \to \mathbb{R}$ defined by

$$V_1^{\mathcal{R}}(\boldsymbol{q}) := \underset{\boldsymbol{r}\in\mathbb{R}^{N+1}}{\mathrm{Max}} \left\{ V_1^M(\boldsymbol{q}-\boldsymbol{r}) + V_1^L(\boldsymbol{r}) : (q^0 - r^0 + V_1^L(\boldsymbol{r}), q^1 - r^1, ..., q^N - r^N) \in \mathcal{R} \right\}. \quad (3)$$

Definition 3, which is drawn from Acerbi and Scandolo (2009), reveals that the liquidity-adjusted value provides an adjustment of portfolio value to rule $\mathcal{R}$ without effectively altering portfolio composition. It answers the following question: Given a certain portfolio, what would time 1 portfolio value be if positions had to be changed in order to fulfill rule $\mathcal{R}$ under liquidity frictions? If the portfolio satisfies rule $\mathcal{R}$, then there is no need to alter existing positions. Hence the portfolio is not exposed to liquidity frictions which entails that the liquidity-adjusted value is equal to the marked-to-market value. In contrast, when rule $\mathcal{R}$ is not fulfilled by the portfolio, then positions have to be changed. Liquidity-adjusted value is then given by the portfolio which among all portfolios satisfying rule $\mathcal{R}$ maximizes value under liquidity frictions. Precisely, given a portfolio $\boldsymbol{q}$, assume it is optimal to liquidate a subportfolio $\boldsymbol{r}^\star$ to fulfill rule $\mathcal{R}$. Subportfolio $\boldsymbol{r}^\star$ is then valued at its liquidation value. As to the remaining portfolio $\boldsymbol{q} - \boldsymbol{r}^\star$, it is valued at its marked-to-market value since this set of positions is unchanged and thus not exposed to liquidity frictions.

For any portfolio $\boldsymbol{q}$ we obtain the following closed-form expression for the time 1 liquidity-adjusted value:

$$V_1^{\mathcal{R}}(\boldsymbol{q}) = q^0 + \sum_{i=1}^{N} S_1^i q^i - \sum_{i=1}^{N} \alpha^i (f^i(q^i))^2, \quad (4)$$

where f^i is defined by

$$f^i(q^i) := \begin{cases} q^i - \tau_i & \text{if } q^i > \tau^i \\ 0 & \text{else} \end{cases} \quad (5)$$

when $\mathcal{R} = \mathcal{R}^S$ and by

$$f^i(q^i) := \begin{cases} q^i - \rho / E_0[\mathrm{VaR}_\gamma(X^i_\delta q^i)] & \text{if } E_0[\mathrm{VaR}_\gamma(X^i_\delta q^i)] > \rho \\ 0 & \text{else.} \end{cases} \tag{6}$$

when $\mathcal{R} = \mathcal{R}^V$. Proof of this result is given in appendix B. Notice that when rule $\mathcal{R}$ is fulfilled by the portfolio, then $f^i(q^i) = 0$ for all i and the liquidity-adjusted value is equal to the marked-to-market value as there is no need to trade the portfolio.

Letting $\boldsymbol{q}$ be any portfolio chosen at time 0, the time 1 liquidity-adjusted value is given by $V_1^{\mathcal{R}}(\boldsymbol{q}) = q^0 \exp(r_f \Delta) + \sum_{i=1}^N S_1^i q^i - \sum_{i=1}^N \alpha^i (f_i(q^i))^2$. In the liquidity-adjusted setting, portfolio return between dates 0 and 1 is then defined by $R_\Delta^{\mathcal{R}}(\boldsymbol{q}) := (V_1^{\mathcal{R}}(\boldsymbol{q}) - V_0^M(\boldsymbol{q}))/V_0^M(\boldsymbol{q})$. It follows that in the liquidity-adjusted setting mean-variance is given by $\mathrm{MV}[R_\Delta^{\mathcal{R}}(\boldsymbol{q})] = E_0[R_\Delta^{\mathcal{R}}(\boldsymbol{q})] - \frac{\theta}{2} V[R_\Delta^{\mathcal{R}}(\boldsymbol{q})]$ with

$$E_0[R_\Delta^{\mathcal{R}}(\boldsymbol{q})] = \frac{q^0(\exp(r_f\Delta) - 1) + \sum_{i=1}^N q^i\left(E_0[S_1^i] - S_0^i\right) - \sum_{i=1}^N \alpha^i \left(f^i(q^i)\right)^2}{q^0 + \sum_{i=1}^N S_0^i q^i}$$

and

$$V[R_\Delta^{\mathcal{R}}(\boldsymbol{q})] \quad = \quad V[R_\Delta^{M}(\boldsymbol{q})]$$

where functions f^i are defined either by equation 5 or by equation 6 depending on the rule that is considered. Notice that portfolio variance in the liquidity-adjusted setting is equal to marked-to-market variance. This results from the liquidity cost term being a deterministic function. Finally, optimal allocation in the liquidity-adjusted setting is given by the portfolio $\boldsymbol{q}$ solution of the problem

$$\underset{\boldsymbol{q} \in \mathbb{R}^{N+1}}{\mathrm{Max}} \mathrm{MV}\left[R_\Delta^{\mathcal{R}}(\boldsymbol{q})\right] \text{ subject to } V_0^M(\boldsymbol{q}) = W_0, \ q^i \geq 0, \ i \in [0, N]. \tag{7}$$

where the constraint $V_0^M(\boldsymbol{q}) = W_0$ reflects the idea that portfolio managers do not face liquidity frictions at time 0 as they are not required to trade at short notice.

3 Comparative Statics

We test in this section the economic relevance of the model developed in the previous section. Precisely, we assess how changes in model parameters value affect optimal allocation under fund size growth in the presence of liquidity frictions. This significates that optimization problem given by expression 7 is repeated for an increasing sequence $W_0^0 < W_0^1 < ... < W_0^n$ of time 0 fund sizes under different parametrizations. Then, for each time 0 fund size, the optimal weight invested in each equity $i \in \{1, ..., N\}$ is computed as $\omega_i^\star = S_0^i \times (q_0^i)^\star / W_0$ and the weight invested in the risk-free asset as $\omega_0^\star = q_0^\star / W_0$. This examination is conducted for both rules. Setup need be specified regarding the VaR rule. We define the VaR rule threshold by $\rho = \kappa W_0$ with $\kappa \in (0,1)$. This significates that the maximum loss per position the portfolio manager is prone to lose at a given confidence level is a certain fixed proportion of time 0 fund size. Table 1 provides the parametrization that is used as benchmark in the sequel.

Parameter	**Symbol**	**Scale**	**Equity 1**	**Equity 2**	**-**
Date 0 MtM Price	S_0	-	10	10	-
Liquidity	α	$\times 10^{-7}$	5	5	-
Drift	μ	$\times 10^{-2}$	2	2	-
Volatility	σ	$\times 10^{-1}$	2	2	-
Correlation	ρ_{12}	-			0
Annual risk-free rate	r	$\times 10^{-2}$			1
Mean-Variance Parameter	θ	$\times 10^{-1}$			5
Management Period	T	$\times 10^{-1}$			5
VaR rule relative threshold	κ	$\times 10^{-2}$			5
Size rule threshold	τ	$\times 1,000$			10
VaR Confidence Level	γ	$\times 10^{-2}$			99

Table 1: **Benchmark parametrization.**

Figure 1 displays optimal allocations in both the marked-to-market and the liquidity-adjusted settings under benchmark parametrization. Precisely, left and right panels both compare marked-to-market allocation to liquidity-adjusted allocations under respectively the VaR rule and the size rule. In each panel the leftmost set of bars displays optimal allocation in the marked-to-market setting while other sets of bars display optimal allocation in the liquidity-adjusted setting for different fund sizes. As there are no liquidity frictions in the marked-to-market setting, optimal proportions remain the same regardless of fund size. This explains that contrarily to the liquidity-adjusted setting, only one set of bars is depicted for the marked-to-market setting.

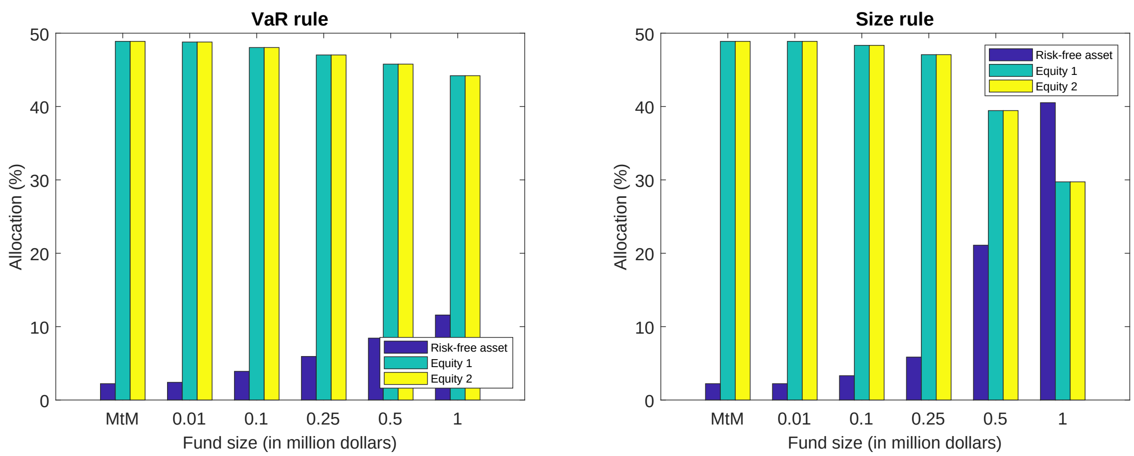

Figure 1: **Optimal allocations under VaR and size rules for benchmark parametrization.** Under both rules the share of wealth invested in the risk-free asset increases with fund size at the expense of wealth invested in equities. Notice that this behavior is smoother under VaR rule than under size rule. Given that the two equities have the same parameters their weights are identical for each fund size.

Given that the risky assets have the same parameters, they are equally weighted in the marked-to-market setting on the one hand and in the liquidity-adjusted setting for both rules and all fund sizes on the other hand. Next, share of wealth invested in the risk-free asset is larger in the liquidity-adjusted setting than in the marked-to-market setting. Further, due to the presence of size-driven liquidity frictions, weight invested in the risky assets diminishes to the benefit of weight invested in the risk-free asset. This can be explained as follows. As fund size increases, size and VaR of risky positions mechanically rise as well. It follows that under both rules the liquidity-adjusted value incorporates the liquidity cost stemming from the liquidation that would be necessary to satisfy the rule. This liquidity cost reduces the return-to-risk feature of risky assets which finally leads to an increase of the allocation towards the risk-free asset. Notice in addition that this behavior is smoother under the VaR rule than under the size rule. This stems from the fact that the VaR rule formulates a risk threshold which is proportional to time 0 fund size while the size rule has a fixed quantity threshold. Consequently, under the VaR rule liquidity-adjustment occurs even for small fund sizes. In opposition, under the size rule liquidity-adjustment starts once fund size attains a certain value. This for instance explains that under the VaR rule optimal allocation for a small \$0.01 million fund differs from marked-to-market allocation while under the size rule allocation is exactly the same.

Next, we investigate how optimal allocation is affected by a rise of equity 2 illiquidity. Precisely, we consider an increase of equity 2 liquidity parameter from 5×10^{-7} to 5×10^{-6}. For this purpose figure 2 provides optimal allocations for both rules under the new parametrization. In addition, optimal weights previously obtained under benchmark parametrization are depicted by thin red bars added onto new weights. First, remark that for both rules and all fund sizes the increase of equity 2 illiquidity has no effect on the weight invested in equity 1. This is pictorially depicted by red bars onto equity 1 bars having the same height than equity 1 bars. By contrast, under both rules the surge of equity 2 illiquidity entails either a reduction or no alteration of the weight of this equity in optimal allocation[1]. Further, the drop of weight invested in equity 2 benefits to the allocation towards the risk-free asset, this latter seeing its proportion enhance with fund size. In a nutshell this significates that following an increase of equity 2 illiquidity, the reduction of weight invested towards this latter equity is not offset by a rise of equity 1 weight (even if more liquid) but by an augmentation of the risk-free weight. This is explained by the fact that increasing allocation towards equity 1 would actually generate additional liquidity frictions while increasing allocation towards the risk-free asset is liquidity frictionless.

[1]Occurences where there is no alteration of equity 2 weight correspond to those where fund size is small under the size rule. As previously explained for the benchmark parametrization case, optimal allocation for small funds under the size rule is the same as in the marked-to-market setting hence an alteration of equity 2 liquidity is inconsequential.

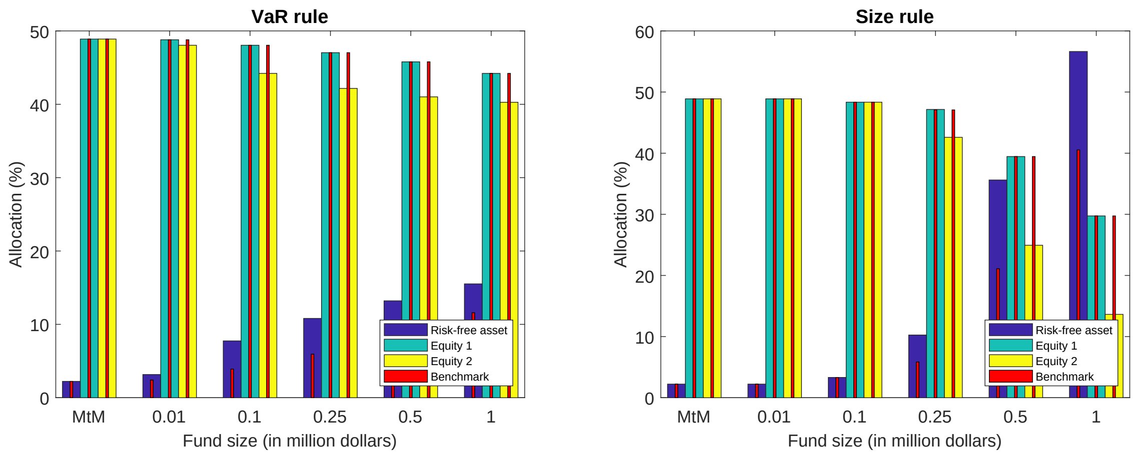

Figure 2: **Change of optimal allocations under equity 2 illiquidity increase.** For each rule and fund size allocation under benchmark parametrization is depicted by a thin red bar. The figure exhibits the change of allocations entailed by an increase of equity 2 liquidity parameter from 5×10^{-7} to 5×10^{-6}. First, for both rules and all fund sizes weight invested in equity 1 is unchanged compared to benchmark parametrization. Next, as equity 2 is more illiquid than under benchmark allocation, weight invested in this asset is reduced in favor of weight invested in the risk-free asset.

We then analyze the effect of an increase of equity 2 volatility parameter from 0.2 to 0.3 on optimal allocations. Figure 3 exhibits optimal allocations under both rules for all fund sizes of interest. One may observe that in both the marked-to-market setting and the liquidity-adjusted setting, the rise of equity 2 volatility results in a reduction of the weight invested in this equity to the benefit of the risk-free asset, weight invested in equity 1 remaining constant. Indeed, as equity 2 has become riskier compared to benchmark allocation, its proportion in total allocation diminishes.

More interesting is to examine optimal allocation behavior following equity 2 volatility surge in the liquidity-adjusted setting. First, under the VaR rule weight of equity 2 remains constant for all fund sizes while that of equity 1 diminishes in favor of the risk-free asset. This behavior is different from that of the benchmark parametrization where weights of both risky assets decrease with fund size in favor of the risk-free asset. This may be explained as follows. As equity 2 is more risky than equity 1, allocation towards the former is such that the VaR rule is fulfilled for this equity for all displayed fund sizes. Hence, no value liquidity-adjustment is entailed for equity 2 positions which explains that weight invested in this equity remains constant and equal to market-to-market weight. One would see the proportion invested in equity 2 decrease for larger fund sizes which are not displayed in figure 3. The same behavior and explanation actually hold for the size rule. Indeed in figure 3 weight invested in equity 2 remains constant up to a \$0.5 million fund size. The difference with the VaR rule is that under the considered parametrization the fund size from which weight invested in equity 2 starts decreasing is smaller. Actually one may observe that for a \$1 million fund weight invested in equity 2 is reduced compared to other depicted fund sizes.

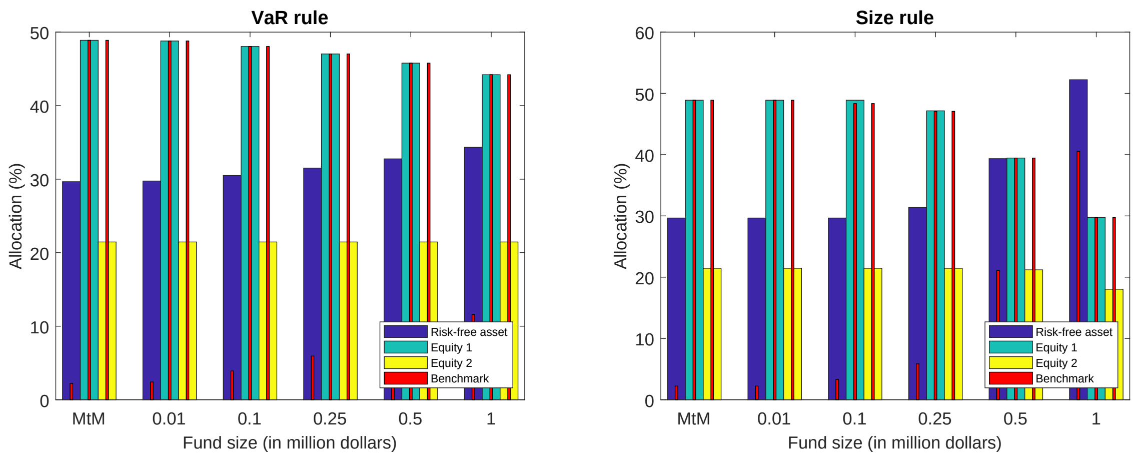

Figure 3: **Change of optimal allocations under equity 2 volatility increase.** The figure depicts the change of optimal allocations entailed by an increase of equity 2 volatility parameter from 0.2 to 0.3. First, this rise of equity 2 volatility is inconsequential for weight invested in equity 1 which remains the same as under benchmark parametrization. Contrariwise, for both rules and all fund sizes, weight invested in equity 2 significantly drops in favor of weight invested in the risk-free asset.

We finally examine the effect of an increase of rules stringency on optimal allocation. Regarding the VaR rule, the increase of severity is expressed through a reduction of the relative risk threshold κ from 0.05 to 0.01. The new risk threshold implies that VaR of each position must remain beneath one percent of time 0 fund value. As for the size rule, the severity enhancement is formulated by a decrease of positions' maximum size from 10,000 shares to 5,000 shares. Figure 4 depicts optimal allocations for both rules after severity rise. Notice that under the VaR rule, weight allocated toward the risk-free asset is larger under the more stringent policy than under the benchmark parametrization. This risk-free weight expansion is made at the expense of weight invested in the set of equities. Under the size rule, this result holds for large funds. However, for small fund sizes, even if severity of the size rule is increased, positions' sizes are still too narrow to attain the size threshold. Hence liquidity-adjusted allocation under the size rule for \$0.01 million and \$0.1 million funds remain unchanged and are equal to the marked-to-market allocation.

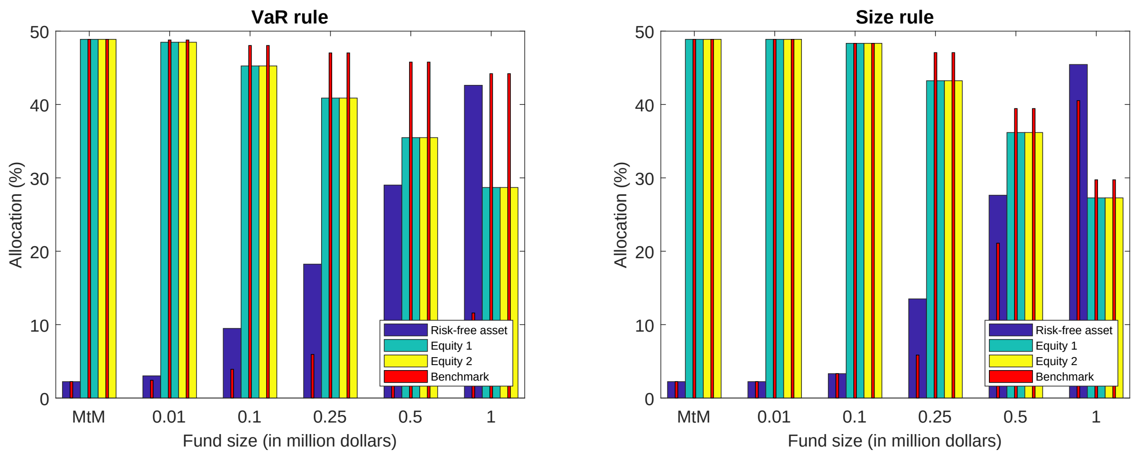

Figure 4: **Change of optimal allocations under rules stringency augmentation.** The figure exhibits the change of optimal allocations entailed by a reduction of the VaR rule relative risk threshold from 0.05 to 0.01 and of the size rule threshold from 10,000 shares to 5,000 shares. For both rules and all fund sizes this increase of severity leads to a rise of wealth proportion invested in the risk-free asset at the expense of wealth proportion invested in the equities.

4 Data

Two datasets are used to estimate parameters of the model. The first one is a high-frequency dataset of trades while the second one is a dataset of daily ask and bid close prices. These two datasets involve 600 equities traded either at NYSE or at NASDAQ. Precisely, the equities on which the study is based are those composing the S&P 600 index as of October 5, 2017. The S&P 600 index is known to gather the small-cap range of companies listed at NYSE and NASDAQ. The data are downloaded from the Thomson-Reuters *Eikon*^{TM} financial analysis solution. More information about this tool is available at: https://financial.thomsonreuters.com/en /products/ tools-applications/trading-investment-tools/eikon-trading-software.html.

The high-frequency trades dataset is used to estimate equities' liquidity parameters only. It consists for the 600 equities of all trades that occured over a six months period ranging from September 15, 2017 to March 15, 2018. Recorded variables are trade price, trade size, and trade time with tick-by-tick granularity. Tick-by-tick granularity is the thinest possible granularity for market data, which means that for a given equity we dispose of the integrality of trades over the period that is considered. Consequently, the total number of trade observations available for the present study (before data processing) amounts to roughly five hundred millions. Notice that the number of trades over the period considered may sensibly differ among equities. It varies from a few thousands tens of thousands of trades for the less traded equities to several millions for equities with high trading activity. Next, the dataset of daily ask and bid prices is used to estimate drift and volatility parameters. As the granularity is thicker than in the first dataset, parameters are estimated over a longer time period. This time period varies from two years of observations for the most recent issued equities to several decades for other stocks. Before estimating model parameters, several processings of the data need be conducted that we detail in the following paragraphs.

First, trades direction need be inferred. Trade direction indicates whether the order that led to trade execution was a sell or a a buy. As is often the case for historical trade databases, trade direction is not provided in the (high-frequency) dataset we examine. We thus use the tick rule (Holthausen *et. al.*, 1987) to infer trade direction from the data. The tick rule has the benefit of relying only on trade prices which is suitable for this study as quotes are not available. The tick rule classifies a trade as a buy if it leads to an increase of the trade price, and as a sell if the trade entails a decrease of the trade price. When the trade price remains the same between two consecutive trades, the trade is classified as the one that precedes.

Then, trades with same timestamp need be distinguished. The timestamp of a trade is the recorded point of time at which this trade occurred. It is displayed as a date and a time of that date. Precision of the timestamp is one second. Due to high-frequency trading, multiple trades may occur within one second. However, trade times accuracy is limited to one second in the data, which makes that all trades that occured within the same second have the same timestamp. We increase time accuracy by considering milliseconds in the study. When several trades occur at the same second, we change trade times according to the following rule: A time increment equal to one second divided by the number of trades is defined; The first trade within the considered second keeps the same timestamp; For the following trades, timestamps are successively increased by the time increment. As an examples, assume that 3 trades are recorded on any given day at 2:16:22 p.m. We then divide one second in three equal time invervals; The first trade is assigned the timestamp 2:16:22.000 p.m.; For the second trade 2:16:22 p.m. is replaced by 2:16:22.333 p.m.; For the third trade we replace 2:16:22 p.m. by 2:16:22.666 p.m..

Finally, we compute daily close mid-prices using daily close ask and bid prices. Daily close mid-prices are then utilized to estimate drift and volatility parameters for all equities which thus do not incorporate market liquidity frictions. Precisely, for each day of the dataset, the daily close mid-price is computed as the average of the daily close ask price and the daily close bid price. This calculation is conducted for all equities of the dataset and for all days within the period that is considered.

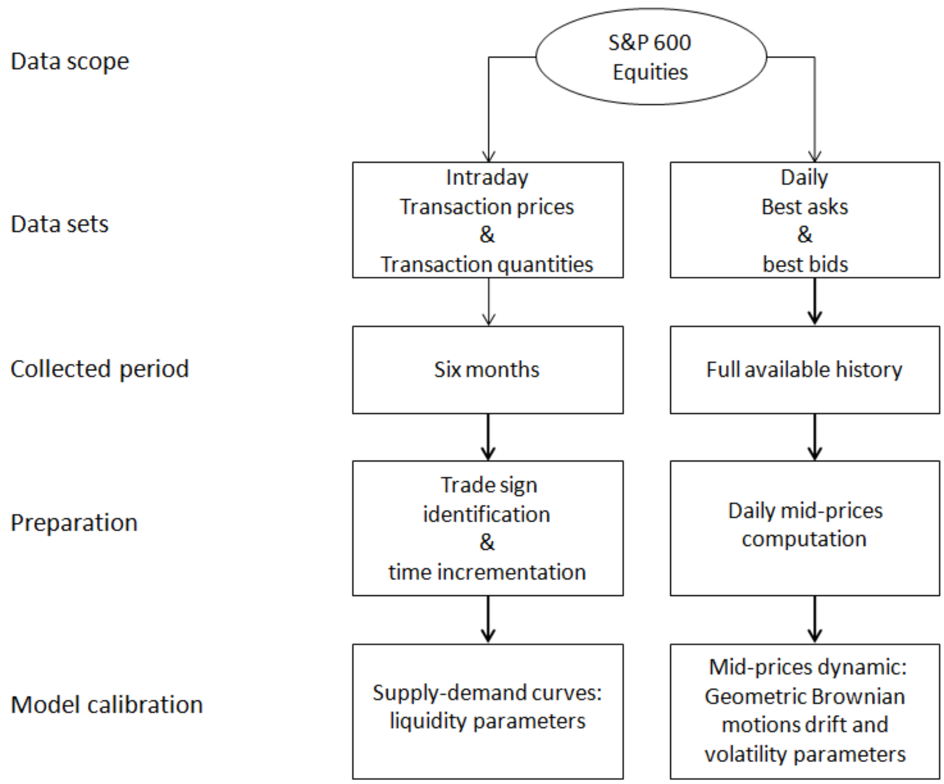

Figure 5: **Data and data preparation.**

All along this paper, it will be useful to consider several S&P 600 representative samples of different sizes that will be utilized to conduct a variety of examinations. After data processing, we have 570 equities among those belonging to the S&P 600 that may be used. We select three subsets of respectively ten, twenty, and fifty percent of total equities in a way that best represents the distribution of market capitalizations of S&P 600 companies. We use systematic sampling (see Fuller (2009) for a reference textbook on sampling techniques) to pick out the equities, which relies on ordering the population according to some criterion and then selecting elements at regular intervals in that population. Precisely, consider an available population of N equities from which only $n < N$ should be retained. Defining $k = N/n$ as the length of the sampling interval, we use market capitalization as the selection criterion in the following manner. All $N = 57$ equities are ranked by increasing market capitalization. That is, the equity with the smallest capitalization is assigned rank 1 while the equity with the largest capitalization gets rank N. A starting rank, denoted i, is drawn randomly from a discrete uniform distribution in $\{1, 2, ..., N\}$. Then, the rank of the first selected equity is given by the nearest unit to $i+k$. The rank of the second selected equity is given by the nearest unit to $i+2k$. This sequence continues until all n equities are determined. Notice that if the rank becomes larger than N (the index of the company with the greatest capitalization), then the algorithm loops to the beginning of the population. Using this procedure, each equity has an equal probability to be picked out.

5 Model Calibration

We present in this section the methodology put forward to calibrate the model of this paper. Estimation of liquidity parameters relies on the high-frequency trades dataset and requires some further modelling that is developped in the first subsection. In contrast, estimation of drift and volatility parameters is based on mid-price series with daily granularity. Output estimates and confidence intervals are provided for the fifty-seven S&P 600 representative equities set.

5.1 Model specification for liquidity parameters estimation

Recall that we previously defined the supply-demand curve as the function $\mathcal{S}(x) = S - \alpha x$. We specify the supply-demand curve in a discrete-time dynamic setting to enable estimation of liquidity parameters. Considering a sequence of $n > 0$ trades, let t_k denote the time at which the k-th trade occurs. Then, given a mid-price S_k and a traded quantity X_k, we define the transaction price by $\mathcal{T}_k := S_k - \alpha X_k$. The mid-price S_k represents the equilibrium price of the asset and is driven by fundamental economic information. The other term of the transaction price, that is $-\alpha X_k$, describes the price deviation from the mid-price entailed by a trade quantity X_k under liquidity frictions. Brown *et al.* (2010) assume that the temporary price impact, which reflects liquidity frictions, only depends on the rate of trading and is independent of the permanent impact. In the same line we model the trade quantity X_k as independent from the mid-price S_k.

Aït-Sahalia et al. (2011) point out that when considering high-frequency data, assigning a drift to price process dynamic becomes irrelevant both economically and statistically. We hence model the mid-price process $(S_k)_{k\in\{1,\dots,n\}}$ as the discretized solution of an arithmetic Brownian motion where drift is set equal to zero. Letting $\Delta t_k := t_{k+1} - t_k$ be the time increment between two consecutive trades, it follows that dynamic of the mid-price is given by $S_{k+1} = S_k + \beta\sqrt{\Delta t_k}\epsilon_1$ where β and ϵ_1 denote respectively volatility of the mid-price and a standard normal random variable. Next, let $\Delta X_k := X_{k+1} - X_k$ denote the quantity increment between two consecutive trades. We model ΔX_k as a normal random variable with mean and standard deviation respectively given by m and s.

Finally, transaction price increments between two consecutive trades need be considered to estimate liquidity parameters. Let $\Delta\mathcal{T}_k := \mathcal{T}_{k+1} - \mathcal{T}_k$ denote the transaction price increment between any two consecutive trades. Under the modelling put forward in the previous paragraphs we end up with the following main equation

$$\Delta\mathcal{T}_k = \beta\sqrt{\Delta t_k}\epsilon_1 - \alpha(m + s\epsilon_2) \tag{8}$$

where ϵ_1 and ϵ_2 are two independent standard normal random variables.

5.2 Liquidity parameters estimation

For each equity in the data, the set of parameters $\{m, s, \alpha, \beta\}$ is estimated according to the following path. First, parameters m and s are estimated by log-likelihood maximization using the model $\Delta X_k = m + s\epsilon_2$ where ϵ_2 is a standard normal random variable. Next, parameters α and β are also estimated by log-likelihood maximization utilizing equation 8 where m and s are replaced by estimates obtained upon the previous step. Notice that for each equity parameters m, s, and β are estimated only in order to get an estimate for α. In the following these parameters are dropped and only parameter α is retained to conduct an empirical analysis.

Quality of estimates $\{m, s, \alpha, \beta\}$ is assessed by computing ninety-five percent confidence intervals. In the case of α and β, these confidence intervals are established by bootstrap as no closed-form expression can be established. Bootstrap methodology is as follows. For each equity 1,000 paths $(\Delta X_i)_{i\in[1,n-1]}$ of trade quantity increments are simulated under estimates $\hat{m}$ and $\hat{s}$ obtained on the historical series. Next, 1,000 paths $(R_{i+1})_{i\in[1,n-1]}$ of returns are simulated under parametrization $\{\hat{\alpha}, \hat{\beta}\}$. Using simulated distributions of $(\alpha^{(k)})_{k\in[1;10,000]}$ and $(\beta^{(k)})_{k\in[1;10,000]}$, ninety-five percent confidence intervals are computed.

Table 3 in appendix C displays estimates as well as ninety-five percent confidence intervals obtained for the set $\{m, s, \alpha, \beta\}$ of parameters belonging to the fifty-seven S&P 600 representative sample. An examination of table 3 enables noticing that parameters estimates always belong to their associated confidence interval. Further, confidence intervals are very narrow which implies that produced estimates are reliable for the sequel of this study.

5.3 Drift and volatility parameters estimation

We estimate in this subsection for each equity parameters μ and σ under the mid-price model given by equation 1. Recall that this set of estimations is conducted on the daily dataset of mid-prices, hence contrarily to liquidity parameters estimation, both drift and volatility parameters may be assessed. Letting t_j indicate a particular trading day, we define the log-return between two consecutive trading days by $R_{j+1} = \log(S(t_{j+1})) - \log(S(t_j))$. The log-return may then be rewritten as $R_{j+1} = (\mu - \sigma^2/2)\Delta + \sigma\sqrt{\Delta}\epsilon$ where $\Delta := t_{j+1} - t_j$ is the time interval between two consecutive trading days and where ϵ is a standard normal random variable. Notice that Δ is now even as we are dealing with daily data and no longer with high-frequency data. Using this latter equation of returns, we estimate parameters μ and σ by log-likelihood maximization.

Estimates for μ and σ are provided by table 5 in appendix C for the fifty-seven S&P 600 representative equities sample. In addition, table 5 compares output estimates to their respective ninety-five percent confidence intervals. An examination of this table enables noticing that parameters estimates always belong to the associated confidence interval. One may also remark that confidence intervals for parameter μ may be relatively wide for certain equities. This is explained by the fact that for some equities the number of observations may be relatively small (recall we excluded equities for which the number of observations is smaller than two years) which reduces accuracy of confidence intervals. In contrast, confidence intervals for parameter σ are always narrow.

6 Empirical Analysis

6.1 Set-up

In the following empirical experiments, a number of portfolio allocations are performed for a six-month time management horizon on March 16, 2018, that is the first trading day following the calibration period. Several subsets of equities belonging to the S&P 600 index are considered. These subsets which are of different sizes are all determined by using the systematic sampling methodology presented previously in order to best represent S&P 600 capitalizations. Subsets that are considered are composed of 57, 114, and 285 equities (which corresponds respectively to ten, twenty, and fifty percent of the 570 S&P 600 equities that we may use). In addition to these different sets of equities, we include a risk-free asset whose interest rate is that of six-month US government bonds, that is 1.91 per cent annually at the date which is retained. Mean-Variance risk aversion parameter is set equal to $\theta = 5$. Value-at-Risk confidence level is equal to ninety-nine per cent as imposed by Basel III regulatory requirements. VaR rule risk threshold ρ is chosen in the following manner. The fund's portfolio manager or the fund's risk team decides of an overall (monetary) risk threshold representing a certain proportion of time 0 fund value. Letting $\tau \in (0, 1)$ being this proportion, overall risk is given by τW_0. Then, if the fund has invested in n different equities, then the risk threshold ρ must satisfy the equality $n\rho = \tau W_0$. In the subsequent experiments parameter τ is fixed equal to one per cent. The rationale for this modelling is that the risk threshold is made dependent on both fund size and the number of invested equities. This way, a larger fund is enclined to undergo a bigger loss per position. Further, if for a fixed fund size the number of investments increases, the risk threshold per position should diminish. On the contrary, the size rule threshold is fixed to 10,000 shares. As previously mentionned, this models the portfolio managers' willingness to avoid holding too large positions in a liquidity management perspective.

In the following it will be useful to assess diversification within the set of risky equities. For this purpose we use a diversification index (DI in the sequel) put forward by Woerheide and Persson (1993) which is the complement of the Herfindahl concentration index. Assume a portfolio is composed of $N+1$ assets where one is the risk-free asset. Letting ω_i denote the weight of asset $i \in \{0, 1, ..., N\}$, each portfolio must satisfy the equality $\sum_{i=0}^{N} \omega_i = 1$. As we wish to assess diversification only among equities, the weight of risky asset $i \in \{1, ..., N\}$ is given by $\tilde{\omega}_i = \omega_i/(1-\omega_0)$. The diversification index is then defined by $\mathrm{DI} = 1 - \sum_{i=1}^{N} \tilde{\omega}_i^2$ where $\mathrm{HI} = \sum_{i=1}^{N} \tilde{\omega}_i^2$ is the Herfindahl index. The diversification index takes values in $(0, 1)$ and the higher its value, the greater portfolio diversification.

6.2 Optimal allocation behavior under fund size increase

In a first experiment we investigate how optimal allocation and *ex-ante* mean-variance behave as time 0 fund size increases. In particular, we compare marked-to-market allocation with liquidity-adjusted allocation for both risk rules. Several subsets of investments opportunities are considered, representing either ten, twenty, or fifty percent of all S&P 600 equities available for this study.

Table 2 provides optimal allocations of wealth between the risk-free asset on the one hand and the set of equities on the other hand. In addition, it exhibits the diversification index as well as *ex-ante* mean-variance under optimal allocations. The diversification index enables synthesizing how diversification evolves as fund size grows. These data are displayed in the marked-to-marked setting and in the liquidity-adjusted settings for fund sizes varying over {\$1 million, \$10 million, \$50 million, \$100 million}.

As a first remark, notice that for both rules and all subsets of equities that are considered, portfolio diversification within equities is always greater in the liquidity-adjusted setting than in the marked-to-market setting. Further, diversification enhances in the former setting as fund size grows. As an illustration, table 2 exhibits that for a fifty-seven equities set the

Setting	MtM	Value-at-Risk rule				Size rule			
Fund Size ($)	-	1 M	10 M	50 M	100 M	1 M	10 M	50 M	100 M
57 equities									
US 6M (%)	0.0000	0.0000	0.0000	0.0000	0.0156	0.0000	0.0000	0.0000	0.0126
Equities (%)	100.0000	100.0000	100.0000	100.0000	99.9844	100.0000	100.0000	100.0000	99.9874
DI	0.7845	0.7964	0.8921	0.9531	0.9593	0.7847	0.8491	0.9472	0.9615
MV	0.1969	0.1915	0.1526	0.0850	0.0409	0.1965	0.1870	0.1252	0.0811
114 equities									
US 6M	0.0000	0.0004	0.0007	0.0011	0.0021	0.0000	0.0000	0.0001	0.0003
Equities	100.0000	99.9996	99.9993	99.9989	99.9979	100.0000	100.0000	99.9999	99.9997
DI	0.9489	0.9539	0.9618	0.9661	0.9716	0.9492	0.9514	0.9640	0.9633
MV	0.1427	0.1362	0.1266	0.0961	0.0657	0.1421	0.1260	0.1067	0.0438
285 equities									
US 6M	0.0003	0.0003	0,0016	0.0021	0.0027	0,0003	0.0000	0.0007	0.0009
Equities	0.9997	0.9997	99,9984	99.9979	99.9973	99.9997	100.0000	99.9993	99.9991
DI	0.9777	0.9783	0.9785	0.9846	0.9862	0,9753	0.9762	0.9770	0,9809
MV	0.1254	0.1265	0,0913	0.0912	0,0753	0,1234	0.0955	0.0912	0,0905

Table 2: **Optimal allocation for several subsets of S&P 600 equities.** In addition to the marked-to-market setting, both the VaR rule and the size rule are displayed by the table. The increase of the diversification index (DI) with fund size reveals an enhancement of diversification among equities in the liquidity-adjusted setting. This diversification enhancement feature is common to both rules and holds for all sets of investment opportunities. Further, the drop of mean-variance (MV) under fund size rise in the liquidity-adjusted setting unveils that liquidity frictions are incorporated in ex-ante mean-variance assessment.

diversification index under the size rule rises from 0.7847 to 0.9615 as fund size increases from \$1 million to \$100 million. This may be explained as follows. As fund size expands, liquidity frictions mechanically strenghten which reduces *ex-ante* mean-variance. Simply scaling up positions' size in a proportional manner without reviewing weights would be suboptimal as it would ignore differences in equities' liquidity. Hence to restrict the negative effect of liquidity frictions on *ex-ante* performance, optimal allocation leads to altering invested weights. Notice in addition that the diversification index in the marked-to-market setting is always very close to the diversication index in the liquidity-adjusted setting for a \$1M fund. The rationale for this observation is that for this fund size liquidity frictions are relatively narrow which in light of the last explanation leads to having quite similar weights.

An other interesting point regards the variation of mean-variance. One may notice that for both rules and for all subsets of equities, *ex-ante* mean-variance reduces as fund size expands. Table 2 for instance shows that for an investment set composed of 285 equities, mean-variance under the VaR rule drops from 0.1265 to 0.0753 as fund size rises from \$1 million to \$100 million. This indicates that in the liquidity-adjusted setting mean-variance accounts for the reduction of expected returns entailed by liquidity frictions. In other words the liquidity-adjusted setting enables modelling *ex-ante* the behavior that fund performance erodes with size. This size-performance relationship is widely corroborated by empirical literature on realized returns (see *e.g.* Grinblatt and Titman, 1989, Chen *et al.*, 2004, or Yan, 2008).

As a last remark, one may have noticed that weight invested in the risk-free asset is almost always close to zero. For instance, the share of wealth invested in the risk-free asset under the size rule when the set of investment opportunities is composed of fifty-seven equities is 0.0126 percent for a \$100 million fund. Given that proportion invested in the risk-free asset is small even in the marked-to-market setting, we put forward the following plausible explanation for this behavior that we challenge in the next experiments. We claim that the relatively large number of equities involved in the different sets of investment opportunities leads to a high level of diversification which distracts the portfolio manager from investing in the risk-free asset.

6.3 Ten-equities case

In order to investigate in greater detail how optimal allocation behaves under fund size increase, we focus in this subsection on a particular case where the set of investment opportunities is restricted to ten equities and the risk-free asset. Precisely, the equities we deal with are the top ten capitalizations of the S&P 600 index. We particularly concentrate on how diversification within the set of equities and weight invested in the risk-free asset both alter with fund size.

Figure 6 exhibits two bar charts, each of those depicting optimal allocations under one of of the risk rules considered in the present study. In each panel the left-most bar depicts optimal allocation in the marked-to-market setting. Then, other bars on the right provide optimal allocation in the liquidity-adjusted setting for fund sizes varying over {\$1 million, \$10 million, \$50 million, \$100 million}. In addition, table 6 in appendix D displays optimal weights for all eleven assets and for each fund size. It also exhibits *ex-ante* mean-variance under optimal allocation as well as portfolios' diversification indexes.

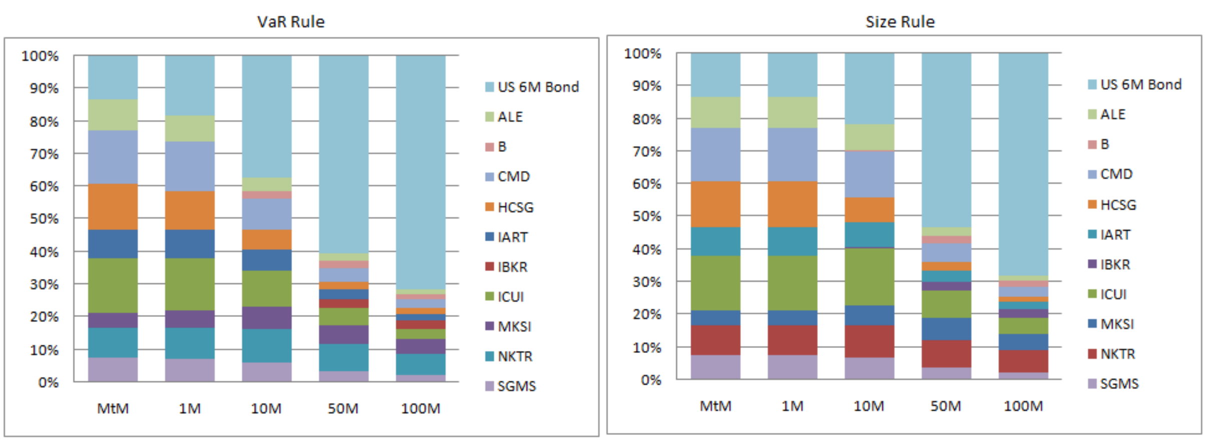

Figure 6: **Optimal allocation behavior under VaR and size rules for a ten-equities set.** For both rules the proportion of wealth invested in the risk-free asset (US 6M Bond) increases with fund size. Further, diversification among equities enhances with fund size under both risk rules. In particular, under the VaR rule eight equities are involved in optimal portfolio composition when fund size is \$1 million. To confine the negative effect of liquidity frictions on *ex-ante* mean-variance, optimal allocation for a \$50 million fund size includes ten equites.

An examination of figure 6 in combination with table 6 enables confirming the previous result that diversification within the set of equities enhances as fund size increases. As an illustration, table 6 shows that under the VaR rule the diversification index rises from 0.8599 to 0.8732 as fund size enlarges from \$1 million to \$10 million. However, a closer look at the ten-equities allocation enables providing more insight about this increase of diversification. Interestingly, notice that even if the set of investment opportunities is constituted of eleven assets, in the marked-to-market setting wealth is invested in only nine of them. This allocation depends on the expected return of the different assets relatively to their respective variance, on correlations with each other, as well as on investor's risk-aversion level. Remark that under the VaR rule, the number of investments moves up as fund size augments. When fund size is small (\$1 million), the number of investments is the same as in the marked-to-market setting (eight equities and the risk free asset) because liquidity frictions are very narrow. In comparison, when fund size is \$50 million, optimal allocation leads to investing in all equities in order to limit the negative effect of liquidity frictions on *ex-ante* mean-variance. Indeed table 6 reveals that proportions of wealth invested in equities B and IBKR rise from zero percent to respectively 2.15 percent and 2.66 percent under the VaR rule. Notice that under the size rule this increase of the number of investments also occurs. To conclude we claim that diversification enhancement among equities is performed not only by reviewing weights of already existing investments, but possibly also by adding new equities from the set of investment opportunities.

We explain this behavior as follows. When fund size is small, liquidity frictions are negligible, hence equities' liquidity features barely matter in liquidity-adjusted portfolio allocation which is thus close to marked-to-market allocation. Thus some equities whose return-to-risk ratio is low are not incorporated in optimal allocation as it would otherwise decrease *ex-ante* mean-variance. However, equities' liquidity becomes more concerning as fund size increases. Due to liquidity frictions, increasing size of already existing positions reduces return-to-risk ratio for these investments. It then becomes interesting for the fund to invest in equities that were left aside from optimal allocation for smaller fund sizes. Indeed, even if these latter equities are endowed with poorer return-to-risk ratios than already invested equities, they may be endowed with better liquidity which makes them attractive for larger fund sizes. Further, this explanation is supported by the empirical finding by Pollet and Wilson (2008) that managers seek new investment opportunities only when liquidity constraints become significant.

An other point highlighted by figure 6 and table 6 regards the allocation towards the risk-free asset. First, as stated in the previous experiment, we find support that the proportion of wealth invested in the risk-free asset is dependent on the number of equities in the set of investment opportunities. Indeed in the present case proportion of wealth invested in the US 6-month bond is 13.95 percent in the marked-to-market setting which is much more significant than values obtained in table 2. More interestingly, figure 6 reveals that in the liquidity-adjusted setting share of wealth invested in the risk-free asset increases with fund size at the expense of the proportion invested in the set of equities. Table 6 for instance indicates that under the VaR rule 18.11 percent of wealth is invested in the US 6-month bond for a $1 million fund while for a $10 million fund this share enhances to 37.10 percent. Recall the explanation for this behavior that we already developed in section 3. Under fund size increase, a myopic marked-to-market allocation simply upscales already existing positions. Hence quantities of all equities held increase together with VaR of risky positions. The liquidity-adjusted value incorporates the liquidity cost stemming from the liquidation that is necessary to satisfy either the VaR rule or the size rule. Hence this liquidity cost reduces the return-to-risk ratio of equities which finally leads to an increase of the allocation towards the risk-free asset.

Further, the behavior highlighted in the previous paragraph may also be analyzed from a risk-management perspective. In order to face future and uncertain liquidity needs, portfolio managers may wish to hold a cushion of cash. However, given a certain fund size if width of the set of investment opportunities is large, then liquidating part of the holdings in order to convert them into cash generates few liquidity frictions and there is no need to hold the risk-free asset. In contrast, if width of investment opportunities is small, then liquidating some holdings at short notice generates strong liquidity frictions and it is then preferable to hold a cushion of cash.

6.4 Out-of-sample experiment

As a next experiment, we conduct a number of out-of-sample simulations to investigate the extent to which liquidity-adjusted allocation leads to a better risk-adjusted performance than marked-to-market allocation. Methodology and setting of the experiment are as follows. For a variety of fund sizes and sets of investment opportunities we take over optimal allocations in both the marked-to-market setting and the liquidity-adjusted setting. Next, for each set of investment opportunities that is considered, we simulate 100,000 times the price vector of equities belonging to this set over a six-month period starting the first day after calibration period. Then, for each draw time 1 price vector is combined with time 0 optimal allocation to obtain final portfolio value.

However, two different scenarios are considered at time 1. In the first scenario which we refer to as *unstressed* scenario, there are no liquidity frictions and the fund may either keep the portfolio unchanged or liquidate it which amounts to the same outcome in terms of portfolio value (marked-to-market). In the second scenario which we refer to as as *stressed* scenario, the fund faces liquidity stress and is required to unwind the portfolio entirely. This entails that the value retrieved at time 1 is the liquidation value. Finally, in both scenarios we compare for each fund size and each set of investment opportunities the risk-adjusted performance of the optimal portfolio under marked-to-market allocation with that of the optimal portfolio under liquidity-adjusted allocation.

Portfolios' performance is assessed applying mean-variance and the Sharpe ratio (Sharpe, 1966; Sharpe, 1994) to distributions of simulated returns. We use the Sharpe ratio in addition to mean-variance as contrarily to this latter measure, the Sharpe ratio is neutral towards investors' preferences[2]. Funds' performance may vary very significantly depending on the allocation that is performed at time 0 (marked-to-market or liquidity-adjusted) and on the scenario that occurs at time 1. This is exhibited by tables 7 (below) and 9 (appendix E) that we comment in the sequel.

[2]Precisely, the Sharpe ratio does not incorporate a risk-aversion parameter and for any given position is simply defined by the average return divided by standard deviation of returns.

Scenario	Unstressed			Stressed		
Allocation	**MtM**	**Size rule**	**VaR r.**	**MtM**	**Size r.**	**VaR r.**
Fund Size: $10 million						
Sharpe ratio	1.4371	1.4344	1.4068	1.0853	1.1851	1.2410
Mean-variance	0.1969	0.1868	0.1525	0.1174	0.1368	0.1472
Fund Size: $50 million						
Sharpe ratio	1.4371	1.2356	1.2048	-1.2710	0.7944	0.8629
Mean-variance	0.1964	0.1250	0.0849	-0.2187	0.0627	0.0744
Fund Size: $100 million						
Sharpe ratio	1.4371	1.1457	1.2026	-7.7654	0.4838	0.7943
Mean-variance	0.1964	0.0810	0.0407	-0.6851	0.0141	0.0351

Figure 7: **Simulated performance statistics in the 57 equities case.** In the unstressed scenario for all displayed fund sizes both the Sharpe ratio and mean-variance are larger under marked-to-market allocation than under liquidity-adjusted allocation (Size rule and VaR rule). This simply shows that in the absence of liquidity frictions at maturity the best time 0 allocation is the marked-to-market allocation. In contrast, in the stressed scenario the Sharpe ratio and mean-variance are greater under liquidity-adjusted allocation than under marked-to-market allocation. In particular, for large funds risk-adjusted performance becomes negative under marked-to-market allocation while it remains positive under liquidity-adjusted allocation. This unveils that a time 0 allocation anticipating a liquidity stress at time 1 offers *ex-post* a better risk-adjusted performance than an allocation which is myopic towards liquidity frictions.

Let us first examine risk-adjusted performance statistics when at time 1 the unstressed scenario occurs. Remark that for all displayed fund sizes, the Sharpe ratio and mean-variance are larger under marked-to-market allocation than under liquidity-adjusted allocation. As an illustration, when fund size is $10 million, the Sharpe ratio and mean-variance are respectively worth 1.4371 and 0.1964 in the marked-to-market setting. In contrast, in the liquidity-adjusted setting under the size rule the Sharpe ratio is worth 1.4344 and mean-variance 0.1929. This lower performance in the liquidity-adjusted setting is explained by the fact that portfolios are constrained by either the size rule or the VaR rule which aim at reducing respectively liquidity and shortfall risk. Then, less risk-taking entails lower returns. Further, the adjustment of time 1 portfolio value to liquidity accounts for liquidity frictions which erodes portfolio performance. Hence, the conjunction of these two elements leads to having smaller risk-adjusted performance outputs under liquidity-adjusted allocation.

In addition, notice that in the unstressed scenario performance spreads between marked-to-market allocation and liquidity-adjusted allocation significantly increase with fund size. For instance, when fund size rises from \$10 million to \$100 million the Sharpe ratio under marked-to-market allocation remains stable at 1.4371 while under size rule allocation its value drops from 1.4344 to 1.1457. As in the liquidity-adjusted setting liquidity frictions become more consequential with fund size augmentation, optimal allocation deviates increasingly from marked-to-market allocation which explains the rise of the performance spread.

Next, we analyze risk-adjusted performance when at time 1 the stress scenario arises. As a first point, remark that in opposition to the unstressed scenario, for any given fund size risk-adjusted performance is higher under liquidity-adjusted allocation than under marked-to-market allocation. For example, for a \$10 million fund size mean-variance under marked-to-market allocation is worth 0.1174 in contrast to 0.1472 under VaR rule allocation. The explanation for this outcome is that liquidity-adjusted allocation chooses at time 0 a portfolio that anticipates a plausible stress scenario at time 1 requiring to liquidate partially positions held.

Further, one may observe that risk-adjusted performance spreads between marked-to-market allocation and liquidity-adjusted allocation raise as fund size expands. Interestingly, for large funds (\$50 million and \$100 million), liquidity-adjusted allocations enable having positive risk-adjusted performance outputs while these latter would be negative under marked-to-market allocation. For instance, as fund size rises to \$50 million, mean-variance under marked-to-market allocation drops to -0.2187 while under VaR rule allocation it keeps a positive value of 0.0744. This latter statement illustrates the benefit of choosing at time 0 a portfolio in a way that accounts for liquidity frictions at time 1.

In addition, we investigate if the stylized facts presented in the previous paragraphs still hold when the set of investment opportunities is extended from 57 to 114 equities. An examination of table 9 in appendix E enables concluding that results obtained for the fifty-seven equities case in the unstressed scenario are still valid when the set of investment opportunities is increased to 114 equities. That is, for all displayed fund sizes marked-to-market allocation leads to a better risk-adjusted performance than liquidity-adjusted allocation. Let us next inspect the stress scenario. First, it is recovered that for large funds liquidity-adjusted allocation leads to greater risk-adjusted performance than marked-to-market allocation. A change however occurs when one considers small funds. In opposition to the 57 equities case, when fund size is small ($10 million) risk-adjusted performance is larger under marked-to-market allocation than under liquidity-adjusted allocation. As an illustration, the Sharpe ratio under marked-to-market allocation is worth 1.1378 against 1.0263 under liquidity-adjusted allocation. This change may be explained as follows. Given that the set of investment opportunities is increased, for an unchanged size the fund on average holds smaller positions. This entails that even in the stress scenario liquidity frictions are relatively narrow. Hence marked-to-market allocation leads to a better risk-adjusted performance.

As a conclusion to this experiment, the major contribution of liquidity-adjusted allocation to portfolio risk-adjusted performance must be emphasized. In an unstressed scenario free of liquidity frictions, the liquidity-adjusted allocation slightly reduces risk-adjusted performance for small funds. Only for very large funds the liquidity-adjusted allocation may noticeably underperform compared to marked-to-market allocation. This is however the counterpart to having higher risk-adjusted performance under liquidity-adjusted allocation in a stress scenario. Indeed in a stress scenario liquidity-adjusted allocation avoids having strongly negative risk-adjusted performance as obtained under marked-to-market allocation that would probably lead funds to bankruptcy.

7 Conclusion

We put forward a model of asset allocation that accounts for market liquidity frictions. The model recovers the empirically documented fact that fund performance diminishes with fund size. It enables prescribing how funds portfolio managers should react as new financial resources enter the fund. Economical relevance of the developed setup is tested through comparative static tests. Finally, we calibrate the model on S&P 600 data to conduct a number of empirical experiments.

We obtain the following main results. First, given a certain fund size, diversification within stocks is enhanced in the liquidity-adjusted setting compared to the setting where market liquidity is not taken into account. Further, diversification in the liquidity-adjusted setting increases with fund size. This latter result significates that liquidity-adjusted allocation alters weights as fund size augments in order to restrain the negative effect of liquidity frictions on *ex-ante* performance. However, diversification is not only obtained by a simple revision of weights in already existing investments. Given a certain set of investment opportunities, empirical experiments show that the model prescribes to increase the number of investments as fund size rises. This result is in line with the empirical finding by Pollet and Wilson (2008) that managers seek new investment opportunities when liquidity constraints become significant. In addition, we obtain that wealth proportion invested in the risk-free asset also rises with fund size.

Outcomes obtained in the present paper demonstrate that investment funds should account for market liquidity when making investment decisions. Especially large funds are exposed to liquidity risk and should revise portfolio allocation under incoming investment capital. Incorporating market liquidity in optimal allocation may not only increase performance compared to the setting where frictions are ignored, but it may also protect against significant capital erosion in a liquidity event turmoil.

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Appendix

Appendix A. Model diagram

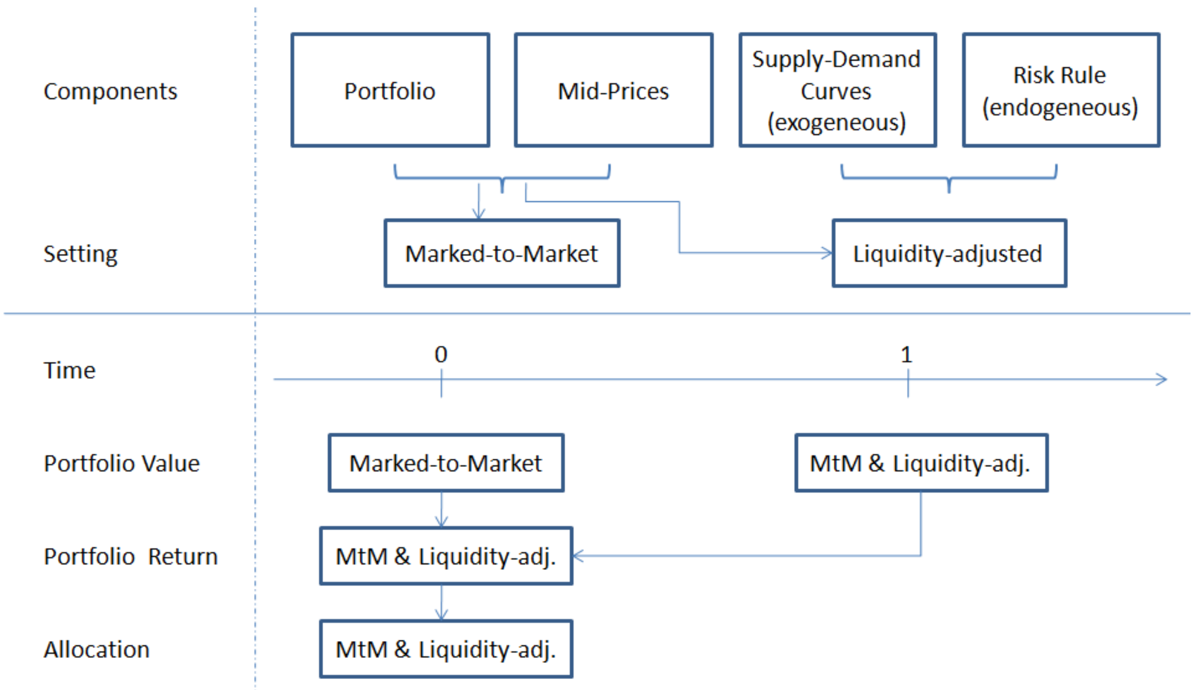

Figure 8: Model Diagram

Appendix B. Derivation of liquidity-adjusted value.

We derive the closed-form expression for $V_1^{\mathcal{R}}$ in the case where $\mathcal{R}$ is the size rule given by $\mathcal{R} = \mathcal{R}^S = \left\{ \boldsymbol{q} \in \mathbb{R}^{N+1} : q_i \leq \tau_i,\ i \in \{1, ..., N\} \right\}$. Optimization problem 3 defining $V_1^{\mathcal{R}}$ may then be rewritten as

$$\underset{\boldsymbol{r} \in \mathbb{R}^{N+1}}{\text{Max}}\, q^0 + \sum_{i=1}^{N} S_1^i (q^i - r^i) + \sum_{i=1}^{N} r^i (S_1^i - \alpha^i r^i) \text{ subject to } q^i - r^i \leq \tau^i, i \in \{1, ..., N\}.$$

We solve this problem using a Lagrangien that we define as

$$L(r^1, ..., r^N, \lambda^1, ..., \lambda^N) := q^0 + \sum_{i=1}^{N} S^i q^i - \sum_{i=1}^{N} \alpha^i (r^i)^2 + \sum_{i=1}^{N} \lambda^i \left[\tau^i - (q^i - r^i) \right]$$

where for $i \in \{1, ..., N\}$ parameters λ^i denothe the Lagrange multipliers. Partial derivatives with respect to r^i and λ^i yield the following system of first order conditions which must be solved:

$$\frac{\partial L}{\partial r^i}(r^1, ..., r^N, \lambda^1, ..., \lambda^N) = -2\alpha^i r^i + \lambda^i = 0 \tag{9}$$

$$\frac{\partial L}{\partial \lambda^i}(r^1, ..., r^N, \lambda^1, ..., \lambda^N) = \tau^i - (q^i - r^i) = 0 \tag{10}$$

Equation 9 yields $r^i = \lambda^i/(2\alpha^i)$ that we replace in equation 10. It follows that optimal value for λ^i is given by $\lambda^{i*} = 2\alpha^i(q^i - \tau^i)$. Next, optimal quantity r^{i*} is obtained by replacing the expression of λ^{i*} in equation 10 which gives the solution $r^{i*} = q^i - \tau^i$. As a final step, replacing r^i by the expression of r^{i*} in the objective function yields the closed-form solution given by equation 4.

Appendix C. Parameters estimates.

	$\hat{\alpha}$ ($\times 10^{-5}$)		$\hat{\beta}$		$\hat{m}$		$\hat{s}$	
Ticker	Estimate	95% C.I.	Estimate	95% C.I.	Estimate	95% C.I.	Estimate	95% C.I.
ABAX	26.1740	(26.0034; 26.3585)	47.6515	(47.2162; 48.1208)	-0.0007	(-0.739302; 0.738093)	109.4840	(108.964887; 110.009560)
ABCB	3.7575	(3.7317; 3.7854)	30.6063	(30.2852; 30.9963)	0.0029	(-2.662283; 2.662732)	362.7955	(360.925042; 364.690394)
ABM	1.3057	(1.2994; 1.3139)	12.5115	(12.3577; 12.6241)	0.0078	(-5.653320; 5.658084)	856.7247	(852.749783; 860.748123)
ADC	5.9289	(5.8918; 5.9789)	15.5196	(15.2591; 15.6763)	0.0021	(-2.527987; 2.533315)	298.0985	(296.322889; 299.901790)
ASNA	0.1103	(0.1098; 0.1109)	2.8515	(2.8173; 2.8757)	-0.0754	(-7.539261; 7.528437)	1.382.4332	(1.377.131617; 1.387.785998)
BANR	0.9278	(0.9209; 0.9345)	6.5964	(6.5469; 6.6484)	0.0008	(-0.939353; 0.939726)	144.7936	(144.133100; 145.461804)
BEAT	54.2440	(54.0220; 54.4729)	23.7761	(22.6960; 24.8508)	-0.0065	(-1.275289; 1.271440)	209.4476	(208.552144; 210.352943)
BGFV	5.1792	(5.1424; 5.2062)	10.3626	(9.9882; 10.6872)	-0.0323	(-4.939196; 4.782161)	652.2757	(648.861455; 655.735494)
BJRI	3.7160	(3.6851; 3.7374)	20.3936	(20.1858; 20.6268)	-0.0429	(-1.471131; 1.385643)	191.8393	(190.836066; 192.856111)
BNED	3.9544	(3.9252; 3.9799)	16.8556	(16.6864; 17.0767)	-0.0751	(-2.856280; 2.705537)	372.4529	(370.499949; 374.432748)
CATM	5.1502	(5.1361; 5.1670)	15.2426	(15.1224; 15.3625)	0.0006	(-0.638628; 0.640387)	155.5862	(155.135694; 156.040082)
CBB	3.0771	(3.0343; 3.1127)	13.6862	(13.3579; 13.9821)	-0.8426	(-7.990843; 6.305408)	502.7310	(497.740754; 507.850228)
CMO	0.3785	(0.3753; 0.3810)	2.1985	(2.1670; 2.2419)	0.3714	(-6.744656; 7.487355)	797.7770	(792.785963; 802.849598)
CPLA	46.3285	(45.8528; 46.7901)	44.8038	(44.0688; 45.7021)	-0.1242	(-1.735351; 1.487079)	141.9172	(140.789323; 143.067981)
CROX	1.1998	(1.1954; 1.2052)	9.8244	(9.7491; 9.9138)	0.0583	(-1.238956; 1.355944)	278.7858	(277.871667; 279.706518)
CRY	8.8893	(8.7059; 9.0149)	10.7262	(10.3941; 11.0578)	-0.8456	(-5.208653; 3.517500)	247.7436	(244.706788; 250.877637)
CRZO	1.2621	(1.2590; 1.2643)	17.7346	(17.6731; 17.8073)	-0.0117	(-0.660289; 0.659785)	299.6163	(299.151300; 300.084718)
CTS	13.8046	(13.4885; 14.0959)	13.8066	(13.3714; 14.2298)	0.3701	(-4.033375; 4.773543)	212.2235	(209.167217; 215.395431)
CUBI	22.0156	(21.7456; 22.2224)	29.9750	(29.4967; 30.5997)	-0.0535	(-2.259839; 2.152802)	207.0077	(205.462468; 208.582737)
CVBF	2.2506	(2.2400; 2.2639)	11.7948	(11.7066; 11.8793)	0.0012	(-1.038609; 1.041631)	193.0418	(192.309983; 193.780922)
CVGW	27.3338	(27.1347; 27.6092)	41.1919	(40.6797; 41.6779)	0.0009	(-0.974645; 0.976885)	117.1384	(116.453515; 117.833462)
DAKT	2.4757	(2.4537; 2.5040)	5.4263	(5.3376; 5.5057)	-0.0277	(-2.125226; 2.069678)	195.9212	(194.452306; 197.418610)
EE	6.3307	(6.2426; 6.4046)	21.2507	(21.0023; 21.5238)	-0.6982	(-3.469547; 3.461243)	317.626180	(315.199719; 320.100633)
EGL	11.2420	(11.1296; 11.3528)	17.8551	(17.5614; 18.0347)	-0.0045	(-1.519009; 1.509917)	155.0212	(153.959600; 156.101403)
ELY	1.5077	(1.4974; 1.5165)	3.9626	(3.9090; 4.0270)	0.0029	(-2.281600; 2.283803)	363.0101	(361.405271; 364.633482)
ERA	10.8047	(10.6534 10.9881)	10.0765	(9.8792; 10.3079)	-0.1597	(-2.701430; 2.381966)	149.3500	(147.579973; 151.174759)
ESE	25.2343	(24.8381 25.5761)	27.4525	(26.9209; 28.0323)	0.0025	(-2.281040; 2.286068)	165.9204	(164.325570; 167.555148)
GTY	7.7476	(7.6666; 7.8157)	8.3011	(8.1808; 8.4731)	0.0023	(-1.364047; 1.368745)	150.1816	(149.223363; 151.155756)
HAE	7.7108	(7.6615; 7.7619)	20.8015	(20.5778; 20.9972)	0.0014	(-1.0758367; 1.078970)	155.2646	(154.507345; 156.031020)

Table 3: **Liquidity parameters estimates based on high-frequency data**

	$\hat{\alpha}$ ($\times 10^{-5}$)		$\hat{\beta}$		$\hat{m}$		$\hat{s}$	
Ticker	Estimate	95% C.I.	Estimate	95% C.I.	Estimate	95% C.I.	Estimate	95% C.I.
HCSG	7.5405	(7.5121; 7.5664)	21.0013	(20.8879; 21.1379)	0.0005	(-0.485532; 0.486289)	116.0842	(115.741870; 116.429043)
IBP	25.2635	(24.9777; 25.4337)	35.4129	(35.0259; 35.8203)	-0.0008	(-1.612890; 1.612890)	173.0935	(171.962576; 174.243572)
INGN	58.3638	(58.0313; 58.6958)	96.1744	(95.2341; 97.2506)	0.0012	(-0.628351; 0.630703)	94.5993	(94.156806; 95.047089)
ITGR	5.0664	(5.0062; 5.1263)	7.6580	(7.5501; 7.7823)	-0.0006	(-2.131379; 2.129969)	187.5392	(186.047873; 189.061176)
KWR	98.8904	(97.5285; 100.1870)	112.6792	(110.0945; 115.1229)	0.0070	(-1.775850; 1.789880)	110.7466	(109.504036; 112.025560)
MNRO	13.5866	(13.4890; 13.6684)	33.3351	(32.9147; 33.7319)	0.0005	(-1.085149; 1.089201)	160.6709	(159.906787; 161.444279)
NPK	30.4159	(29.7388; 31.0573)	42.3600	(41.4396; 43.4341)	0.1690	(-3.248805; 3.586609)	176.8659	(174.490836; 179.324718)
NSIT	13.1612	(13.0905; 13.2227)	23.6076	(23.3565; 23.8744)	-0.0010	(-0.811134; 0.809248)	117.3594	(116.790006; 117.935786)
OFG	3.0996	(3.0560; 3.1422)	9.3426	(9.1107; 9.5432)	0.2950	(-5.719442; 6.309143)	383.8712	(379.677993; 388.184029)
OSIS	18.8406	(18.7624; 18.9070)	47.5281	(47.1641; 48.0826)	0.0005	(-0.719842; 0.720993)	154.7580	(154.250799; 155.269612)
PETS	9.8798	(9.8364; 9.9238)	36.0981	(35.6937; 36.4440)	0.2112	(-0.673253; 1.096015)	164.4222	(163.799844; 165.050894)
POWI	26.1685	(25.9898; 26.3478)	47.2170	(46.6782; 47.6168)	0.0076	(-0.888480; 0.903627)	113.3984	(112.769347; 114.036561)
QNST	3.9153	(3.9007; 3.9318)	11.5784	(11.4538; 11.7187)	0.0064	(-1.124949; 1.137406)	199.7342	(198.938375; 200.538090)
RAVN	13.6021	(13.4567; 13.7267)	27.1395	(26.7665; 27.4645)	0.0053	(-1.434663; 1.445293)	144.2843	(143.275129; 145.311595)
RMBS	0.9794	(0.9762; 0.9831)	6.7692	(6.7188; 6.8341)	0.0014	(-1.171194; 1.173758)	236.2633	(235.437918; 237.096032)
RYAM	3.2107	(3.1846 3.2286)	8.6324	(8.5231; 8.7398)	-0.0103	(-1.947928; 1.932810)	277.2858	(275.922291; 278.666382)
SANM	3.9361	(3.9191 3.9591)	26.6973	(26.3910; 26.9014)	0.0013	(-1.351599; 1.353026)	226.6271	(225.675829; 227.588275)
SEDG	6.2044	(6.1710; 6.2304)	33.3001	(33.0503; 33.5989)	-0.0094	(-1.383031; 1.383031)	267.8895	(266.916295; 268.872178)
SEM	2.4863	(2.4632; 2.5117)	12.9928	(12.8458; 13.1919)	-0.0240	(-5.117174; 5.106511)	505.1276	(501.545251; 508.774603)
SNCR	1.3074	(1.3032; 1.3109)	9.2945	(9.2157; 9.3699)	-0.0167	(-1.768443; 1.636307)	383.8824	(382.683326; 385.090820)
SPTN	7.6189	(7.5755; 7.6648)	17.8554	(17.6612; 18.0670)	0.0414	(-0.864208; 0.947177)	128.6176	(127.981117; 129.261958)
TREE	142.3832	(141.6102; 142.9146)	322.0759	(320.3243; 324.1641)	0.0002	(-0.401353; 0.400378)	82.8231	(82.540861; 83.107765)
UVV	16.9960	(16.8134; 17.1531)	19.5577	(19.1389; 19.9447)	-0.0058	(-2.228442; 2.215744)	201.2155	(199.659774; 202.802359)
VAC	44.9202	(44.6284; 45.2374)	76.0250	(74.9207; 76.8805)	-0.0045	(-1.227515; 1.218638)	131.4592	(130.601618; 132.331328)
VRTS	114.3663	(112.8022; 115.8991)	108.9855	(106.7731; 111.0727)	-0.1848	(-1.506091; 1.136430)	94.4142	(93.491644; 95.360276)
WABC	24.6049	(24.3764; 24.8550)	38.8746	(38.3163; 39.3807)	-0.0013	(-0.921883; 0.919241)	103.3892	(102.743463; 104.045345)
WRLD	65.8404	(65.4133; 66.3685)	135.0614	(133.2754; 137.7460)	-0.0476	(-0.962828; 0.867760)	100.7132	(100.071367; 101.365800)
XOXO	8.9156	(8.7909; 9.0571)	9.1864	(9.0047; 9.3646)	-0.1696	(-2.663480; 2.324617)	176.4430	(174.701775; 178.229063)

Table 4: **Liquidity parameters estimates based on high-frequency data (continued)**

	$\hat{\mu}$		$\hat{\sigma}$			$\hat{\mu}$		$\hat{\sigma}$	
Ticker	Estimate	95% C.I.	Estimate	95% C.I.	Ticker	Estimate	95% C.I.	Estimate	95% C.I.
ABAX	0.322949	(0.119973; 0.526095)	0.527788	(0.518938; 0.537031	HCSG	0.240053	(0.110857; 0.369242)	0.337253	(0.331621; 0.343132)
ABCB	0.214247	(0.067268; 0.361338)	0.364859	(0.358457; 0.371559)	IBP	0.533130	(0.172118; 0.894354)	0.370858	(0.355635; 0.387846)
ABM	0.171001	(0.085120; 0.256868)	0.265230	(0.261475; 0.269126)	INGN	0.671563	(0.290496; 1.053005)	0.391346	(0.375274; 0.409281)
ADC	0.114380	(-0.000252; 0.229093)	0.285196	(0.280203; 0.290421)	ITGR	0.198802	(0.031760; 0.366083)	0.354872	(0.347622; 0.362518)
ASNA	0.211426	(0.047698; 0.375180)	0.427343	(0.420204; 0.434794)	KWR	0.257801	(0.101414; 0.414345)	0.369501	(0.362700; 0.376642)
BANR	0.186041	(0.016520; 0.355668)	0.407991	(0.400614; 0.415724)	MNRO	0.212826	(0.096405; 0.329236)	0.303458	(0.298385; 0.308758)
BEAT	0.359723	(0.029300; 0.690320)	0.531231	(0.517020; 0.546479)	NPK	0.111331	(0.035843; 0.186980)	0.233436	(0.230134; 0.236867)
BGFV	0.225681	(-0.025744; 0.477205)	0.507004	(0.496111; 0.518521)	NSIT	0.438349	(0.222384; 0.654372)	0.528414	(0.519013; 0.538259)
BJRI	0.323549	(0.124078; 0.523154)	0.469516	(0.460841; 0.478621)	OFG	0.259203	(0.108022; 0.410452)	0.390290	(0.383699; 0.397172)
BNED	-0.023747	(-0.579488; 0.532586)	0.458768	(0.435644; 0.485275)	OSIS	0.318490	(0.088121; 0.548914)	0.529022	(0.519010; 0.539540)
CATM	0.435826	(0.097750; 0.774090)	0.550241	(0.535693; 0.565834)	PETS	0.264448	(0.058878; 0.470023)	0.392940	(0.384050; 0.402370)
CBB	0.209184	(0.081602; 0.336858)	0.393723	(0.388142; 0.399513)	POWI	0.223022	(-0.001626; 0.447790)	0.371498	(0.361827; 0.381854)
CMO	0.064561	(-0.044344; 0.173465)	0.316161	(0.311404; 0.321108)	QNST	0.211147	(-0.113972; 0.536430)	0.470337	(0.456407; 0.485396)
CPLA	0.259466	(0.032720; 0.486279)	0.388357	(0.378583; 0.398795)	RAVN	0.259053	(0.129928; 0.388161)	0.330606	(0.324979; 0.336484)
CROX	0.461355	(0.106474; 0.816559)	0.626879	(0.611563; 0.643206)	RMBS	0.521779	(0.221284; 0.822505)	0.697575	(0.684509; 0.711295)
CRY	0.366656	(0.135564; 0.597879)	0.533982	(0.523937; 0.544535)	RYAM	0.221519	(-0.315396; 0.759101)	0.528553	(0.505963; 0.553890)
CRZO	0.348708	(0.093545; 0.603942)	0.588152	(0.577058; 0.599798)	SANM	0.438669	(0.199830; 0.677693)	0.606466	(0.596057; 0.617346)
CTS	0.198219	(0.074125; 0.322422)	0.383466	(0.378036; 0.389097)	SEDG	0.660292	(-0.007305; 1.328814)	0.584477	(0.556589; 0.616206)
CUBI	0.244459	(0.008900; 0.480082)	0.263280	(0.253304; 0.274314)	SEM	0.214345	(-0.028838; 0.457703)	0.359827	(0.349400; 0.371085)
CVBF	0.242126	(0.094244; 0.390006)	0.345034	(0.338605; 0.351782)	SNCR	0.392959	(0.064098; 0.722175)	0.573404	(0.559220; 0.588545)
CVGW	0.311520	(0.123763; 0.499312)	0.377565	(0.369432; 0.386166)	SPTN	0.203479	(-0.008512; 0.415540)	0.452179	(0.442979; 0.461873)
DAKT	0.337374	(0.149681; 0.525198)	0.468327	(0.460152; 0.476882)	TREE	0.649220	(0.296061; 1.002588)	0.555531	(0.540347; 0.571835)
EE	0.176001	(0.068736; 0.283397)	0.256500	(0.251833; 0.261397)	UVV	0.140945	(0.056716; 0.225240)	0.260364	(0.256679; 0.264187)
EGL	0.279969	(-0.044527; 0.604695)	0.393561	(0.379760; 0.408704)	VAC	0.421396	(0.203966; 0.638911)	0.278617	(0.269350; 0.288740)
ELY	0.243955	(0.090124; 0.397865)	0.399475	(0.392769; 0.406479)	VRTS	0.474434	(0.180126; 0.768987)	0.454124	(0.441481; 0.467725)
ERA	0.052458	(-0.330698; 0.436008)	0.442352	(0.426096; 0.460281)	WABC	0.138768	(0.033761; 0.243872)	0.260908	(0.256333; 0.265694)
ESE	0.256521	(0.126912; 0.386135)	0.345173	(0.339523; 0.351071)	WRLD	0.364630	(0.196331; 0.533054)	0.438834	(0.431496; 0.446497)
GTY	0.151084	(0.055639; 0.246577)	0.294899	(0.290722; 0.299229)	XOXO	0.280074	(0.043430; 0.516781)	0.432847	(0.422627; 0.443720)
HAE	0.189517	(0.075057; 0.304072)	0.301891	(0.296900; 0.307104)					

Table 5: **Drift and volatility estimates and 95% confidence intervals for S&P 600 representative equities sample.**

Appendix D. Optimal allocations in the ten-equities case.

Setting	MtM	VaR rule				Size rule			
Fund Size ($)	-	1M	10 M	50 M	100 M	1 M	10 M	50 M	100 M
US 6M	13.47	18.11	37.10	60.64	71.30	13.47	21.51	53.35	68.11
ALE	9.50	8.13	4.39	2.22	1.69	9.51	8.07	2.60	1.52
B	0.00	0.00	2.37	2.15	1.71	0.00	0.22	2.42	1.68
CMD	16.33	15.24	9.47	4.09	2.63	16.33	14.25	5.46	3.25
HCSG	13.86	11.92	5.84	2.34	1.57	13.86	7.66	2.59	1.52
IART	8.73	8.50	6.44	3.25	2.19	8.73	7.74	3.65	2.31
IBKR	0.00	0.00	0.00	2.66	2.46	0.00	0.00	2.60	2.44
ICUI	16.78	15.97	11.03	4.97	3.19	16.78	17.68	8.50	5.02
MKSI	4.70	5.34	7.17	5.95	4.49	4.70	6.22	6.65	5.00
NKTR	9.18	9.42	10.05	8.31	6.44	9.18	9.85	8.54	6.72
SGMS	7.45	7.37	6.15	3.40	2.32	7.45	6.81	3.65	2.42
MV	0.0813	0.0789	0.0661	0.0465	0.0372	0.0813	0.0793	0.0568	0.0428
DI	0.8573	0.8599	0.8732	0.8739	0.8746	0.8573	0.8574	0.8751	0.8766

Table 6: **Optimal allocation behavior under VaR and size rules for a ten-equities set.** For both rules the proportion of wealth invested in the risk-free asset (US 6M Bond) increases with fund size. Further. rise of the diversification index exhibits the enhancement of portfolio diversification within risky assets. Further, diversification among equities enhances with fund size under both risk rules. Finally, *ex-ante* mean-variance reduces in the liquidity-adjusted setting due to size-driven liquidity frictions.

Appendix E. Out-of-sample simulations: 114 equities case.

Scenario	**Unstressed**			**Stressed**		
Allocation	**MtM**	**Size rule**	**VaR r.**	**MtM**	**Size r.**	**VaR r.**
Fund Size: $10 million						
Sharpe ratio	1.2338	1.1226	1.1347	1.1378	1.0263	1.0596
Mean-variance	0.1426	0.1259	0.1266	0.1234	0.1020	0.1074
Fund Size: $50 million						
Sharpe ratio	1.2338	1.1629	1.1846	0.7171	0.8077	0.9282
Mean-variance	0.1427	0.1065	0.0960	0.0502	0.0652	0.0852
Fund Size: $100 million						
Sharpe ratio	1.2338	1.0774	1.0579	0.0922	0.5243	0.6633
Mean-variance	0.1427	0.0438	0.0655	-0.0424	0.0219	0.0421

Figure 9: **Simulated performance statistics in the 114 equities case.**

Cross asset

INVESTMENT STRATEGY

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