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Bond Portfolio Optimisation and Mixed Integer Programming

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Abstract

While portfolio optimisation is commonplace in equities, it is more complex in the fixed-income space, partly because of trading lot sizes. Implementing the portfolio composition by converting weights into holdings is easy for equities due to small lot sizes. If we consider bonds, the difference between the model and the resulting portfolio may be significant, especially when the notional amount at stake is not large. In previous research, we have approched bond-index tracking by using genetic algorithms. In this research, we explore the use of mixed-integer optimisation techniques and show that it is straightforward to implement in either corporate or government bond portfolios for any given portfolio size, subscriptions, redemptions, and portfolio rebalancing. In particular, we prove that linear constraints implying one to several bond characteristics or cardinality constraints can be handled without prior stability or convergence tests.

Keywords: Bond indexation, portfolio replication, branch-and-bound, discrete optimisation, integer programming, constraint handling, duration, credit risk.

JEL classification: C61, G11, G12.

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Mohamed Ben Slimane is the Head of Fixed Income Quant Portfolio Strategy within Amundi Institute. On top of the long-term research subjects of the team, Mohamed specialises in designing new thinking for problems which have not yet been solved within investment platforms for active or passive fixed income management from the upper stages down to prototyping and testing.

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He started his career at Lyxor AM as Portfolio Manager in 2012. From 2012 to 2016, he managed developed and emerging market funds on equity and fixed-income passive investments. He managed essentially through index funds and ETFs but also worked on dedicated mandates, customized passive solutions, and employee saving funds. He priced and traded equities and fixed-income instruments and helped setting-up processes and tools in order to replicate most indices, including FX-Hedged products. On another side, he participated in developing and maintaining equity and fixed-income basket trading tool, portfolio-sampling optimizer, and day-to-day management tools for dedicated and more "complex" investment strategies. In 2016, He was dedicated almost full-time to fixed-income portfolios. He was also among the major stakeholder of fixed-income PCFs generation tools and AP Portal (tools dedicated to ETFs).

During the acquisition of Lyxor by Amundi, he was very active in defining the target operating model and in migrating fixed-income ex-Lyxor funds to ALTO (Amundi IT system) ensuring all processes go smoothly during the transition period.

Ghassen holds an engineering degree from CentraleSupelec, former Ecole Centrale Paris (2011). He also holds a quantitative master degree (MASEF) from Paris Dauphine University (2012) and is a CFA Charterholder (2022).

1 Introduction

Fixed-income exchange-traded funds are a kind of exchange-traded fund (ETF) that invests in bonds such as treasuries, corporate bonds, or convertibles[1]. Similarly to bond mutual funds, they hold a portfolio of bonds and offer different strategies and holding periods. Besides the reduced costs for investors, bond ETFs provide exposure to the bond market with the ease and transparency of stock trading. Unlike individual bonds or bond mutual funds[2], they trade throughout the day on major stock exchanges and even in times of distress.

There exists active fixed income ETFs, but the majority of them are passive. For the latter, the main objective is to minimize active risk compared to a reference index and exhibit the same risk/return metrics. Those metrics can include various risk factors such as active weights, modified duration, yield to maturity (or yield to worst if the bond embeds an option), and duration-times-spread (DTS). They are computed either globally or across sectors, issuers, maturity pillars, bonds, and ratings. The target is to minimize active exposure to duration and credit risks while maximizing the portfolio liquidity compared to its benchmark: duration, credit, and liquidity being the main bond price drivers. As to the first, interest rates move depending on the state of the economy. If interest rates rise, yields on outstanding bonds decline relative to new bond issues that will consequently be paying higher coupons. This is referred to as duration risk. Second, corporate bonds are part of the business environment. The capacity of the issuing firm to be successful and be able to respect its debt obligations is another determinant of prices. This is referred to as credit risk. Third, the functioning of the bond market itself plays a role. For corporate bonds in particular the difficulties to trade have a considerable impact on the value of a bond. This is liquidity risk.

ETF's liquidity on exchange and in the secondary market is essential to display tight spreads and reduce the total cost of ownership for investors. It also participates in enhancing the attractiveness of some ETFs compared to others. Consequently, fixed-income ETF portfolio managers may have to comply with market-makers' constraints. Indeed, market makers create ETF units by delivering a basket of underlying securities to the ETF issuer in exchange for a block of ETF units with the same market value. In a redemption process, the market maker exchanges ETF units with the ETF provider for an equivalent basket of underlying securities from the ETF. Since authorized participant market makers (AP) already own bonds inventory to hedge intra-day exposure to ETF, they may be "axed". They are more interested to including in their wish list some bonds of the portfolio because they do not need to buy them in the market to provide the basket to the portfolio manager for the creation or redemption. In this case, AP may then provide fund managers with their constraints on the bonds to be bought or sold. These constraints, referred to hereafter as axes constraints, may impose, if possible, that bonds are part of an authorized list $\mathcal{A}$ and that traded quantities do not exceed an allowed quantity per bond.

Investment guidelines may impose constraints on the sector, country, currency, rating, or issuer by setting a maximum deviation threshold or an exclusion process. The deviation can be set at a global level. For instance, a global constraint could be no more than 5 bps in deviation per sector. The deviation can be set when calculating a contribution to risk. For example, no more than 1.7 bps of deviation in the contribution to duration per

[1] They can marginally invest in future contract and use FX swap and FX forwards for hedged share classes.

[2] Individual bonds are sold over the counter. Bond mutual funds are traded in the best case at one price per day.

maturity pillar. In ESG-oriented funds, constraints on ESG scores and carbon metrics are mandatory. They may set a minimal investment weight in green or sustainable bonds. Last but not least, constraints on turnover or of the number of trades can add up.

Ben Slimane (2021) describes how to implement genetic algorithms in the context of bond index tracking and shows that they can be an industrial solution for passive bond management even though the entry cost to be familiar with these algorithms is high. Indeed, the stability and convergence of investment solutions require thousands of tests before proposing automation to fund managers. In this paper, we use mixed-integer programming instead through the branch-and-bound or cutting planes algorithms to handle additional objectives and constraints on the fly. After showing in Section Two that bond-index tracking can be solved by means of mixed-integer linear programming, we tackle in Section Three the theory behind it. In the last section, we describe the tool used by ETF portfolio managers to help them build and rebalance passive index funds through some examples for different scenarios and different types of objective functions and constraints.

2 Bond-index tracking is a MILP

In Ben Slimane (2021), the genetic-proposed solution matches three risk metrics per sector[3]: modified duration per maturity bucket, duration-times-spread[4], and weight. The solution applies penalties for liquidity and axes constraints. The liquidity is proxied by the product of the clean price and the liquidity score developed by Ben Slimane and De Jong (2017). The lower this value, the more liquid the bond. The axes component is the number of changes in the quantities of bonds that do not belong to axes $\mathcal{A}$. The challenge of handling discrete quantities of bonds has dictated the recourse to genetic algorithms. Bonds, unlike equities, cannot be traded generally at one unit of quantity[5]. Each bond has its minimum tradable amount, under which one cannot buy or sell a quantity of that bond (for example, EUR 100 000). Above this minimum tradable amount, lot size is the incremental nominal amount traded (for example, EUR 1 000). The quantity of a chosen bond i can be written as

$$Q_i = MT_i + l_i LT_i \tag{1}$$

where MT_i and LT_i are the minimum tradable amount and the lot size of bond i, and l_i is a natural number.

After thousands of runs using different self-adaptive or deterministic mechanisms, four intrinsic parameters of the genetic algorithms[6] were calibrated to lower the number of transactions, maximize the number of bonds on the axis and minimize the uninvested amount of subscriptions or redemptions. In the same framework of sector matching, considering additional metrics to match or constraints to satisfy suppose to recalibrate the four parameters. What about matching globally or differently the risk metrics? One may match credit risk at the sector level and simultaneously track the duration risk at the portfolio level. Testing all combinations of risk metrics and constraints to find the best genetic parameters might be time-consuming and might not even lead to optimal solutions in reasonable times. Consequently, we must consider off-the-shelf algorithms to manage tracking objectives and constraints.

[3] Or country if the reference portfolio is a government bond index.

[4] When the reference portfolio bears credit risk. DTS measures the systematic exposure to credit risk by quantifying sensitivity to a shift in the yield spread(Ben Dor *et al.*, 2007)

[5] Tables 10 on page 23 and 11 on page 24 display the breakdowns for the minimum lot size and minimum tradable amount in the MSCI ACWI Index and the Global Aggregate ex-ABS Index.

[6] Population size, termination criteria, crossover rate, and mutation rate

All the metrics we track are $\mathcal{L}_1$-norm risk measures. Any active risk measure, that relates to a metric M (e.g. MD) with regard a feature f (e.g. sector or maturity bucket), can be written as:

$$\mathcal{R}_M \left(x \mid b, f\right) = \left| \sum_{i \in b} \left(x_i - b_i\right) M_i \delta_i \right|$$

where $\delta_i = 1$ if Bond i has the feature f, otherwise 0. The objective function is to minimise a linear combination of active risk measures. The combination may be equally weighted or weighted to emphasise some of the risk components. All the constraints we verify are linear (e.g. the sum of weights) or ruled by an $\mathcal{L}_1$-norm (e.g. the absolute deviation per issuer). Transforming the weights into quantities leads to linear or $\mathcal{L}_1$-norm equations, as shown in appendix A.2.1 on page 21. These quantities are functions of integer variables as seen earlier in Equation (1). All of the above are arguments for the use of Mixed Integer Linear Programming (MILP). If the $\mathcal{L}_2$-norm were to be considered, Mixed Integer Quadratic Programming (MIQP) would be the solution. However, since bond indices contain thousands of securities[7], we reach here the numerical limits of quadratic programming[8]. Indeed, according to Roncalli (2023), quadratic programming algorithms are efficient when the dimension of the problem is relatively small, say, when $n \leq 5000$. The point is not the convergence of the algorithm, but the manipulation of the Hessian matrix Q of the quadratic form where n^2 floating-point numbers are stored.

3 Mixed Integer Linear Programming algorithms

A Mixed Integer Linear Programming (MILP) problem is a mathematical optimisation in which some or all the variables are integers, and the objective function and the constraints are linear. A MILP problem has the following form:

$$\begin{aligned} x^\star &= \arg\min \mathbf{c}^T x \\ & \text{s.t.} \quad \left\{ \begin{array}{l} \mathbf{M}x \leq \mathbf{m} \\ x \geq \mathbf{0} \\ \#\left\{i : x_i \in \mathbb{N}\right\} \geq 1 \end{array} \right. \end{aligned} \tag{2}$$

where $\mathbf{c} \in \mathbb{R}^n, \mathbf{m} \in \mathbb{R}^k$ are vectors and $M \in \mathbb{R}^{k \times n}$ is a matrix.

$\mathcal{L}_1$-norm functions are not linear. Thanks to the absolute value trick[9], we transform them into linear functions by introducing intermediate real variables. The dimension of the problem is however augmented by the number of these new variables.

Integer variables make an optimisation problem non-convex therefore, harder to solve than standard linear programming problems. Memory and solution time may exponentially rise as we add more integer variables. Many combinations of integer values are verified, and each combination requires the solution of a standard linear problem (LP). We generally solve MILP in two steps. First, the problem is made continuous by removing the integral constraint on variables. The new "relaxed" problem is convex and is solved in a polynomial time. Then divide-and-conquer algorithms take over to perform a systematic and eventually

[7] As of 31/03/2023, the Bloomberg Global Aggregate Total Return Index has 29 222 securities.

[8] Alternatively, we can subdivide the universe according to bond categories (corporate, government bonds, supra, and agencies) for aggregate indices and repeat optimisation for each group separately to reduce the dimension of each sub-problem.

[9] See appendix A.2.2 on page 21

exhaustive search. In the following subsections, we expose three algorithms used in MILP and illustrate them with simple examples for ease of comprehension.

3.1 Branch-and-Bound

Branch-and-Bound (*BnB*) is a tree search method proposed by Land and Doig (1960), proven to reach the optimal solution by systematically exploring the set of possible solutions. The basic idea is to recursively divide the initial problem into smaller subproblems using a branching strategy. At each step, the algorithm divides the search space into at least two smaller subspaces to explore. It then evaluates each subspace to determine if it is possible to find an optimal solution there. If an optimal solution is found in a subspace, the algorithm sets it as the best-known solution. Otherwise, the algorithm does not explore that subspace any further and moves to another subspace. *BnB* also uses a bounding technique to reduce the search space by eliminating subspaces that cannot contain an optimal solution. Each branch is checked against upper and lower estimated bounds on the optimal solution and is discarded if it cannot produce a better solution than the best one found so far. This technique helps to avoid wasting time exploring unpromising subspaces.

When we apply *BnB* to an integer programming problem, we use it in conjunction with the standard non-integer approach. Hereafter a simple maximization problem solved with *BnB*:

$$\begin{aligned} (x^\star, y^\star) \quad &= \quad \arg\max f(x,y) = 5x + 8y \\ \text{s.t.} \quad &\begin{cases} 5x + 9y \leq 45 \\ x + y \leq 6 \\ x, y \in \mathbb{N} \end{cases} \end{aligned} \tag{3}$$

The resolution of a problem with *BnB* is described as a search through a tree, where the root node corresponds to the original problem, and each other node corresponds to a subproblem of the original problem. Given a node N of the tree, the children of N are subproblems derived from N by imposing a new single constraint for each subproblem. Thus, the descendants of N are the subproblems, which satisfy the same constraints as N and additionally a number of others.

We begin the resolution of Problem (3) by relaxing the integer constraints. The problem becomes then a LP without integer constraints. The relaxed solutions are then $x_{rel}^{(1)} = 2.25$ and $y_{rel}^{(1)} = 3.75$, for which $f(x_{rel}^{(1)}, y_{rel}^{(1)}) = 41.25$. The first node contains the relaxed and the rounded-down solutions ($x_{rd}^{(1)} = 2$ and $y_{rd}^{(1)} = 3$). An upper and lower bounds[10] are associated with this node, $U^{(1)} = 41.25$ and $L^{(1)} = 34$. The lower bound is the value of f at the rounded-down solution. The optimal integer solution will be between these two bounds.

The first step in *BnB* is to create two solution subsets from the relaxed solution. According to Achterberg *et al.* (2005), the easiest and most common branching strategy incorporated into general MIP solvers is the branching on a variable. It consists of choosing a variable $\bar{x}_i$ with fractional part in the non-integer solution and adding the constraints $x_i \geq \lceil \bar{x}_i \rceil$ and $x_i \leq \lfloor \bar{x}_i \rfloor$. In our case, we choose to retrieve the variable with the highest fractional part. It is y as $\{x_{rel}^{(1)}\} = 0.25$ and $\{y_{rel}^{(1)}\} = 0.75$. We then branch on variable y.

[10] When the problem minimises a function, relaxed solutions are rounded up, and upper and lower bounds are reversed.

The y search space ($\mathbb{N}$) is divided into $y \leq 3$ and $y \geq 4$. We solve the relaxed LP (4) with the new constraint added at each of these nodes.

$$(x^\star, y^\star) = \arg\max f(x,y) = 5x + 8y \quad \text{s.t.} \begin{cases} 5x + 9y \leq 45 \\ x + y \leq 6 \\ x, y \geq 0 \\ y \leq 3 \end{cases} \quad \text{and} \quad (x^\star, y^\star) = \arg\max f(x,y) = 5x + 8y \quad \text{s.t.} \begin{cases} 5x + 9y \leq 45 \\ x + y \leq 6 \\ x, y \geq 0 \\ y \geq 4 \end{cases} \tag{4}$$

We fill in nodes 2 and 3 with their bounds: $x_{rel}^{(2)} = 3$ and $y_{rel}^{(2)} = 3$, $U^{(2)} = 39$ and $x_{rel}^{(3)} = 1.8$ and $y_{rel}^{(3)} = 4$, $U^{(3)} = 41$. The lower bounds $L^{(2)}$ and $L^{(3)}$ are both equal to $U^{(2)} = 39$ because the relaxed solution of node 2 is totally integer and $U^{(2)} > L^{(1)}$. Next, We repeat in node 3 the steps previously performed.

Table 1: *BnB* different nodes

Node N	Parent Node	Additional constraint	Relaxed sol. $x_{rel}^{(N)}$	Relaxed sol. $y_{rel}^{(N)}$	Rounded sol. $x_{rd}^{(N)}$	Rounded sol. $y_{rd}^{(N)}$	$U^{(N)}$	$L^{(N)}$	Node Sol.
1			2.25	3.75	2	3	41.25	34	1
2	1	$y \leq 3$	3	3	3	3	39	39	2
3	1	$y \geq 4$	1.8	4	1	4	41	39	2
4	3	$x \leq 1$	1	4.44	1	4	40.55	39	2
5	3	$x \geq 2$						39	2
6	4	$y \leq 4$	1	4	1	4	37	39	2
7	4	$y \geq 5$	0	5	0	5	40	40	7

Table 1 shows the different steps performed to reach the integer solution. We report for each node, its parent node, the added constraint besides those applied to its parent nodes, the relaxed and rounded-down solutions. The upper and lower bounds, and the node where the current integer solution can be found are shown in the last columns. At node 5, the problem is infeasible. Node 7 shows the optimal integer solution as $U^{(7)} = L^{(7)}$. Thus, it is not possible to achieve any higher value by further branching from this node.

3.2 Cutting plane algorithm

A cutting plane, proposed by Gomory (1960), is an inequality constraint added to the LP, having the following properties: (1) It is not satisfied by the non-integer optimal solution of the old LP, and (2) all the feasible integer solutions of the old LP remain feasible in the new LP. Hence, the plane cuts off part of the feasible region which does not contain any integer feasible solution. The cutting process is repeated until the optimal solution found is also integer. This algorithm is mainly used when the constraints set is too large or when the inequality constraints are insufficient to yield an integer solution. A cutting plane can be derived from different sources, such as the problem geometry or the structure of the objective function. It can also be computed as a by-product of the simplex algorithm for solving LPs. We illustrate this last point with Problem 3.

In Table 2, we report the simplex tableau for the resolution of Problem 3. The optimal solution is the same as above $(x^\star, y^\star) = (\frac{9}{4}, \frac{15}{4})$. To build a cutting plane, we pick usually

Table 2: Problem 3 associated simplex tableau

Base	Variables		Slack variables		RHS
	x	y	s_1	s_2	
Initial Table					
s_1	5	9	1	0	45
s_2	1	1	0	1	6
	-5	-8	0	0	0
First iteration					
y	$\frac{5}{9}$	1	$\frac{5}{9}$	0	5
s_2	$\frac{4}{9}$	0	$-\frac{1}{9}$	1	1
z	$-\frac{5}{9}$	0	$\frac{8}{9}$	0	40
Second iteration					
y	0	1	$\frac{1}{4}$	$-\frac{5}{4}$	$\frac{15}{4}$
x	1	0	$-\frac{1}{4}$	$\frac{9}{4}$	$\frac{9}{4}$
z	0	0	$\frac{3}{4}$	$\frac{5}{4}$	$\frac{165}{4}$

a row from the last iteration with the largest fractional of RHS. So let's consider the first row. We have:

$$\frac{1}{4}s_1 - \frac{5}{4}s_2 = \frac{15}{4} \tag{5}$$

If we separate all coeffcients into integer and fractional parts and keep the fractional parts on the left, Equation (5) becomes

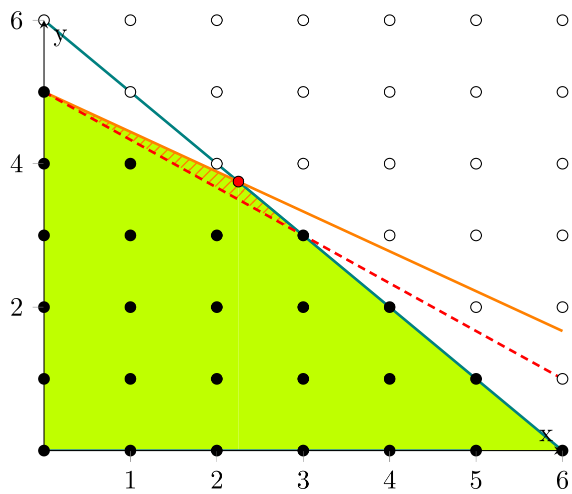
$$\frac{1}{4}s_1 + \frac{3}{4}s_2 - \frac{3}{4} = 3 + 2s_2 \tag{6}$$

If all the variables are integers, the right-hand side of Equation (6) is integer, and so must be the left-hand side (LHS). Since $s_1, s_2 \geq 0$, so $\frac{1}{4}s_1 + \frac{3}{4}s_2 \geq 0$ then LHS $\geq -\frac{3}{4}$. Hence, LHS must be at least 0. Using the definitions of the slack variables in the LHS equation, we deduce the cutting plane as function of x and y:

$$\begin{aligned} \frac{1}{4}s_1 + \frac{3}{4}s_2 - \frac{3}{4} \geq 0 &\Rightarrow \frac{1}{4}(45 - 5x - 9y)_1 + \frac{3}{4}(6 - x - y) - \frac{3}{4} \geq 0 \\ &\Rightarrow 2x + 3y \leq 15 \end{aligned} \tag{7}$$

Figure 1 gives a geometric illustration of our problem. We plot the space of feasible solutions in green and emphasize the non-integer solution with a red dot. The search space is delimited by the green and orange lines, associated with the two constraints of the problem and the first quadrant as variables are positive. The hatched area is the area that contains the non-integer solution $(x^\star, y^\star)$, cut off by the cutting plane from the feasible region. We verify that all possible integer solutions (black dots) before applying the cutting remain possible after the cutting.

Figure 1: Geometric illustration of cutting planes



Next, we introduce a new slack variable $s_3 = \frac{1}{4}s_1 + \frac{3}{4}s_2 - \frac{3}{4}$ and we solve a new LP problem (8). The objective function[11] and the two first constraints are retrieved from Table 2, to which we add the cutting plane constraint.

$$
\begin{aligned}
(x^\star, y^\star, s_1^\star, s_2^\star, s_3^\star) \quad &= \quad \arg\min \frac{3}{4}s_1 + \frac{5}{4}s_2 \\
\text{s.t.} \quad & \begin{cases} y + \frac{1}{4}s_1 - \frac{5}{4}s_2 = \frac{15}{4} \\ x - \frac{1}{4}s_1 + \frac{9}{4}s_2 = \frac{9}{4} \\ \frac{1}{4}s_1 + \frac{3}{4}s_2 - s_3 = \frac{3}{4} \end{cases}
\end{aligned}
\tag{8}
$$

We report in Table 3 the results of the iterations using the simplex method. The optimal solution has integer coordinates $(x = 0, y = 5, s_1 = 0, s_2 = 1, s_3 = 0)$ and is equal to the solution found by the *BnB* algorithm.

3.3 Branch-and-Cut

Branch-and-Cut (*BnC*) is a variant of Branch-and-Bound, first suggested by Padberg and Rinaldi (1991) to solve the Travelling Salesman problem. It uses the same heuristic in the branching step but uses cutting planes when bounding to tighten the bounds on the optimal solution in each branch of the search tree. The rationale is to significantly reduce the branches to be explored to improve the algorithm performance. Using the same example as above, we add the constraint (7) of the cutting plane to the relaxed LP (4). Table 4 shows that the integer solution is reached in only three iterations, compared to the seven iterations of the *BnB* method. Balas *et al.* (1996) argued that applying cutting planes in a branch-and-cut framework provides a powerful algorithm for solving general MILP problems.

When the cutting plane is applied at the top node of the search tree, *BnC* becomes Cut-and-Branch (*CnB*). In our case, the solution is reached in only 1 iteration using (*CnB*).

[11] The canonical form is a minimization

Table 3: Problem 8 associated simplex tableau

Basis	Variables					RHS
	x	y	s_1	s_2	s_3	
Initial table						
y	0	1	$\frac{1}{4}$	$-\frac{5}{4}$	0	$\frac{15}{4}$
x	1	0	$-\frac{1}{4}$	$\frac{9}{4}$	0	$\frac{9}{4}$
s_3	0	0	$\frac{1}{4}$	$\frac{3}{4}$	1	$\frac{3}{4}$
z	0	0	$\frac{3}{4}$	$\frac{5}{4}$	0	0
Last iteration						
y	$\frac{2}{3}$	1	0	0	$\frac{1}{3}$	5
s_2	$\frac{1}{3}$	0	0	1	$-\frac{1}{3}$	1
s_1	-1	0	1	0	-3	0
z	$\frac{1}{3}$	0	0	1	$\frac{8}{3}$	$\frac{5}{4}$

Table 4: *BnC* different nodes

Node N	Parent Node	Additional constraints	Relaxed sol.		Rounded sol.		$U^{(N)}$	$L^{(N)}$	Node Sol.
			$x_{rel}^{(N)}$	$y_{rel}^{(N)}$	$x_{rd}^{(N)}$	$y_{rd}^{(N)}$			
1			2.25	3.75	2	3	41.25	34	1
2	1	$y \leq 3$ + Cut.	3	3	3	3	39	39	2
3	1	$y \geq 4$ + Cut.	0	5	0	5	40	40	3

4 Illustrations

We make use of the FICO Xpress optimiser to solve bond-tracking problems. This solver is part of The FICO Xpress Optimisation Suite, widely used in academia and industry, such as production scheduling, transportation, supply chain management, telecommunications, finance, and personnel planning. The most popular application of the FICO Xpress optimiser is for Mixed Integer Programs (MIP). Usually, and depending on how hard the problem is, the optimiser solves MIP in four stages: (1) Presolving, (2) Solving the initial LP relaxation, (3) Heuristics and (4) Cutting, and (5) Branch-and-bound tree search. Presolving reduces the problem's initial size by dropping unnecessary constraints and variables. It is often the most crucial part, as the reduction in size can cause a significant difference in solution time. Heuristics are rules applied on every node or each k nodes during the tree search to help find a feasible solution. They can be either simple rounding of the continuous solution, local searches, or a combination of these.

Besides the techniques discussed above, the solver uses sophisticated branch variable selection techniques such as pseudo costs[12] or strong branching[13]. Finally, the Xpress optimiser is multi-threaded and utilizes all computer cores. Parallel processing is an attractive way to speed optimisation as different nodes in the MIP tree search can be handled independently.

Table 5 illustrates the time in seconds for Xpress and an open-source solver CBC[14] to solve the same optimisation problem. We present here 4 cases of optimisations in a euro corporate and a global government bond portfolios with different AUM and the same simulated size in two different directions. Xpress is more efficient[15] whether we simulate subscription or redemption in a broad or a small portfolio. With both solvers, the level of difficulty decreases with the portfolio size. All other things being equal, redemption requires more time than subscription because of additional no short-selling and minimum tradable quantity holding constraints.

Table 5: Differences between FICO Xpress and CBC

Portfolio	**Type**	**AUM (M €)**	**Simulated Size (M €)**	**# of bonds in universe**	**Solve time (seconds)**	
					Xpress	CBC
Euro Corp.	Subscription	1 100	10	2 631	0.19	10.32
Euro Corp.	Redemption	1 100	10	1 735	0.19	31.69
Global Gov	Subscription	55	10	996	2.29	89.59
Global Gov	Redemption	55	10	955	12.76	> 2 500

For the corporate portfolio, we seek to minimize, with regard to the reference portfolio, the differences in DTS, weight, and yield at the sector level and the modified duration per each couple of sector and maturity pillar under the constraints of Table 6. Global constraints refer to those checked at the portfolio level. Constraints of absolute deviations or contribution to MD or DTS deviations are verified at the indicated level.

[12]Pseudo cost is an estimated cost of moving a variable either up or down by a unit amount. This cost is multiplied by the distance a variable has to move to reach the next integer value, which provides an estimated change in the objective function.

[13]Strong branching involves testing which candidate variable improves the objective function the most before branching on them.

[14]**CO**IN-OR **B**ranch and **C**ut is an active open-source project from **CO**mputational **IN**frastructure for **O**peration **R**esearch.

[15]FICO Xpress runs on a dedicated server whereas CBC runs on a local machine.

Table 6: Euro Corporate constraints

Type	Constraint	Limit	Maximum	
			Subscription	Redemption
Global	DTS (bps)	2.72	2.72	2.23
	MD (Yr × 100)	1.75	0.24	1.75
Absolute deviation	Sector 1 (%)	0.15	0.11	0.14
	Sector 2 (%)	0.21	0.15	0.19
	Issuer (%)	0.16	0.16	0.16
Contrib. to MD deviation	Sector 1 (Yr × 100)	1.05	0.70	1.02
	Sector 2 (Yr × 100)	2.39	1.49	1.82
	Issuer (Yr × 100)	1.28	1.07	1.23
	Pillar (Yr × 100)	3.93	2.15	3.69
	Pillar × Country (Yr × 100)	2.06	1.04	2.00
Contrib. to DTS deviation	Sector 1 (Yr × 100)	6.02	5.86	5.11
	Sector 2 (Yr × 100)	8.66	7.64	6.64

For instance, no more than 0.16% of drift is acceptable per issuer. Besides these restrictions, any solution must satisfy trade nominals multiple of each bond minimum tradables with at least EUR 50 000 per bond and leave at most EUR 50 000 of cash uninvested. To avoid concentration, a maximum of 3% is set per issuer of traded weight after adjustment. The maximum column reports the maximum drift for each item after optimisations. Except for DTS constraints, the optimiser seems to have less freedom when dealing with redemption. The maximum drifts, even within the lines, are often higher than the ones obtained for the subscription.

Table 7: Euro Corporate optimisation results

Metric	Subscription	Redemption
Total number of variables	**2 631**	**3 464**
of which integer variables	2 631	1 732
of which binary variables	0	1 732
Inequality constraints	**4 299**	**8 596**
of which positivity	2 631	3 464
of which binary-integer association	0	3 464
of which set-up constraints	1 668	1 668
Number of trades	**57**	**76**

Table 7 shows the difference in set-up between the two optimisations where the total number of variables and inequality constraints are reported. The total number of involved variables regarding subscription equals the number of bonds in the universe. It is double that number for redemptions[16] as a binary variable is attached to each integer variable. A binary variable is a variable with only two values (0 or 1). It ensures that the minimum remaining quantity after sale is zero or the minimum tradable. Let q_i and t_i be the current quantity and the minimum tradable of Bond i. Let x_i[17] be the sought multiplier of t_i and

[16] The number of integer variables is below the number of bonds in the universe. A pre-process is performed to avoid under-allocating issuers (when selling) and over-allocating issuers (when purchasing) with regard the reference portfolio.

[17] We constraint $x_i \geq 0$

b_i a binary variable. x_i and b_i satisfy the following inequalities:

$$\begin{cases} x_i \leq \dfrac{q_i}{t_i} - b_i \\ x_i \geq (1 - b_i)\dfrac{q_i}{t_i} \end{cases} \tag{9}$$

Indeed, when no quantity remains ($b_i = 0$), Equation (9) give $x_i = \dfrac{q_i}{t_i}$ as a solution. When a quantity should remain ($b_i = 1$), we obtain the condition $x_i \leq \dfrac{q_i}{t_i} - 1$. Equation (9) adds as a result 3 464 constraints to redemption optimisation as two constraints are associated with each integer variable. Only 1 668 constraints are common to both optimisations if we exclude positivity and Equation (9) constraints.

Table 8: Global Govies constraints

Type	**Constraint**	**Limit**	**Maximum**	
			Subscription	Redemption
Trades	Number	100	100	100
Global	MD (Yr × 100)	0.80	0.79	0.28
Absolute deviation	Country (%)	0.10	0.10	0.09
	Currency (%)	0.10	0.10	0.09
	Bond (%)	0.50	0.50	0.39
Contrib. to MD deviation	Country (Yr × 100)	0.73	0.72	0.62
	Pillar (Yr × 100)	0.28	0.26	0.28
	Pillar × Country (Yr × 100)	0.29	0.24	0.27

Regarding the global government bond portfolio, we minimize, with regard to the reference portfolio, the differences in weight and yield at the portfolio level and the modified duration per each couple of country and maturity pillar. Table 8 gives the additional constraints. Here, no more DTS or sector constraints but currency and country constraints. Generally, the minimum tradable amounts associated with government bonds are too low compared to corporate bonds. For instance, German government bonds can be traded at EUR 0.01. In our optimisation, we set a minimum tradable amount of at least 5 000 and restrict the number of trades not to exceed 100. This last constraint requires another set of binary variables.

Let c_i be a binary attached to x_i. The underlying idea is to count using c_i the number of non-null x_i. Let P_i and Nav be the dirty price of Bond i and the portfolio net asset value. x_i and c_i satisfy the following inequalities:

$$\begin{cases} P_i t_i x_i \leq c_i Nav \\ \sum_i c_i = 100 \end{cases} \tag{10}$$

In Equation (10), the first constraint indicates that the mark-to-market of Bond i cannot exceed the net asset value of the portfolio. c_i equals 0 implies that x_i is 0, and x_i above 0 entails c_i equals 1. However, c_i equals 1 does not force x_i to be non-null. Hence, the equality constraint on the sum over c_i limits the number of trades to no more[18] than 100.

Considering constraints of Equations (9) and (10) lead to double and triple subscription and redemption variables, as shown in Table 9. From 2 489 and 4 089 inequality constraints, only 925 and 790 are related to the set-up if we exclude those of the number of trades. If we exclude those of the absolute deviation by bond, 143 constraints appear to be in common in the two optimisations.

Table 9: Global Govies optimisation results

Metric	Subscription	Redemption
Total number of variables	**1 564**	**1 941**
of which integer variables	782	647
of which binary variables	782	1 294
Equality constraints	**1**	**1**
Inequality constraints	**2 489**	**4 025**
of which positivity	782	1 294
of which binary-integer association	0	1 294
of which number of trades constraints	782	647
of which other set-up constraints	925	790
Number of trades	**100**	**100**

5 Conclusion

Compared to alternative methods such as genetic algorithms, the interest in using the mixed-integer optimisation technique is that it can be easily modified to change the objective functions or the applied constraints. No preliminary tests are needed here to assess the convergence of the investment solutions. Nevertheless, depending on the portfolio structure and constraints tightness, fund managers may relax some constraints if they make the problem infeasible to guarantee the existence of at least one solution reachable by the optimiser.

Another key element of the success of the mixed-integer algorithm is its scalability to the level of portfolio nominal: it is suitable for both small and large portfolios, albeit with a difference in resolution time. From our point of view, bond optimisation through MI is an industrial solution for passive bond management with a low entry cost to be familiar with, handle any additional requirements, and maintain.

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[18] To obtain exact 100 trades, one must add to equation (10) the condition $x_i \geq (c_i - 1 + \epsilon) \cdot \frac{Nav}{P_i t_i}$ where ϵ is very small number. Hence, $c_i = 1$ implies that $x_i \geq \frac{Nav}{P_i t_i}\epsilon$, which is above 0.

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A Appendix

A.1 Notations

We use the following notations:

- b is the reference (or benchmark) bond portfolio;
- x is a portfolio solution candidate;
- x^0 is the current bond portfolio;
- DTS_i, MD_i, YLD_i, LTP_i and P_i are the duration-times-spread price, modified duration, yield, liquidity-times-price, and the dirty price of Bond i;
- $\Delta Q_i(x) = Q_i(x) - Q_i\left(x^0\right)$ is the change in quantity of Bond i in Portfolio x with respect to its quantity in the current portfolio x^0;
- $\mathcal{C}ash(x)$ is the amount of cash of Portfolio x;
- $\text{NAV}(x)$ is the net asset value of Portfolio x:

$$\text{NAV}(x) = \sum_{i=1}^{n_x} P_i \cdot Q_i(x) + \mathcal{C}ash(x)$$

- $\mathcal{A}djustment = \text{NAV}(x) - \text{NAV}\left(x^0\right)$ is the amount of adjustment;
- x_i is the weight of Bond i in Portfolio x:

$$x_i = \frac{P_i \cdot Q_i(x)}{\text{NAV}(x)}$$

- $\mathcal{R}_M\left(x \mid b\right)$ is the global active risk measure related to metric M:

$$\mathcal{R}_M\left(x \mid b\right) = \left| \sum_{i \in b} (x_i - b_i) M_i \right|$$

- $\mathcal{R}_M\left(x \mid b, f\right)$ is the active risk measure related to metric M with regard feature f:

$$\mathcal{R}_M\left(x \mid b, f\right) = \left| \sum_{i \in b} (x_i - b_i) M_i \delta_i \right|$$

  where $\delta_i = 1$ when Bond i has the feature f otherwise 0.

A.2 Mathematical results

A.2.1 Risk measures: From weights to quantities

Globally, the active risk related to a metric[19] M is equal to:

$$\begin{aligned}
\mathcal{R}_M\left(x \mid b\right) &= \left|\sum_{i\in b}(x_i - b_i)\cdot M_i\right| = \left|\sum_{i\in b}\frac{P_i\cdot Q_i\left(x\right)\cdot M_i}{\mathrm{NAV}\left(x\right)} - b_iM_i\right| \\
&= \left|\sum_{i\in b}\frac{P_i\cdot\Big(Q_i\left(x^0\right)+\Delta Q_i\left(x\right)\Big)\cdot M_i}{\mathrm{NAV}\left(x\right)} - b_iM_i\right| \\
&= \left|\sum_{i\in b}\frac{P_i\cdot\Delta Q_i\left(x\right)\cdot M_i}{\mathrm{NAV}\left(x\right)} - \sum_{i\in b}\left(b_i - \frac{P_i\cdot Q_i\left(x^0\right)}{\mathrm{NAV}\left(x\right)}\right)\cdot M_i\right| \\
&= \left|\mathbf{A_M}^T\Delta Q - c_M\right|
\end{aligned}$$

where ΔQ designs the vector of changed quantities, $\mathbf{A_M} = \left(\dfrac{P_i\cdot M_i}{\mathrm{NAV}\left(x\right)}\right)$ is a vector element of $\mathbb{R}^n$ and $c_M = \sum_{i\in b}\left(b_i - \dfrac{P_i\cdot Q_i\left(x^0\right)}{\mathrm{NAV}\left(x\right)}\right)\cdot M_i$ is constant[20] depending on the reference and the initial portfolios.

We obtain similar formula when considering a standalone feature (for example sector, currency, rating, sector X currency, ...). We have:

$$\mathcal{R}_M\left(x \mid b, f\right) = \left|\sum_{i\in b}\left(x_i - b_i\right)\cdot M_i\cdot\delta_i\right| = \left|\mathbf{A_M(f)}^T\Delta Q - c_M(f)\right|$$

where $\mathbf{A_M(f)} = \left(\dfrac{P_i\cdot M_i\cdot\delta_i}{\mathrm{NAV}\left(x\right)}\right)$ is a vector element of $\mathbb{R}^n$, $\delta_i = 1$ when Bond i has the feature br otherwise 0. $c_M(f) = \sum_{i\in b}\left(b_i - \dfrac{P_i\cdot Q_i\left(x^0\right)}{\mathrm{NAV}\left(x\right)}\right)\cdot M_i\cdot\delta_i$ is constant of feature f depending on the reference and the initial portfolios.

A.2.2 From $\mathcal{L}_1$-norm to linear functions

Suppose we track a risk metric M at the portfolio level by minimizing $\mathcal{R}_M\left(x \mid b\right)$. In Appendix A.2.1, we show that the active risk measure can be written as function of the vector of changed quantities ΔQ. We have: $\mathcal{R}_M\left(x \mid b\right) = \left|\mathbf{A_M}^T\Delta Q - c_M\right|$

[19] For instance, the metric could be any weightable metric: MD, MD per maturity bucket, DTS, or yield. When the considered metric is the weight, $M_i = 1$

[20] Once we set the *Adjustment*, $\mathrm{NAV}\left(x\right)$ is constant

$$
\begin{aligned}
\text{Minimizing } \mathcal{R}_M\left(x \mid b\right) \quad &\Longleftrightarrow \quad \text{Minimizing } \tau \text{ s.t. } \left|\mathbf{A_M}^T \Delta Q - c_M\right| \leq \tau \\
&\Longleftrightarrow \quad \begin{cases} \text{Minimizing } \tau \\ \qquad\qquad \text{s.t.} \begin{cases} A_M{}^T \Delta Q - c_M \leq \tau \\ A_M{}^T \Delta Q - c_M \geq -\tau \end{cases} \end{cases} \\
&\Longleftrightarrow \quad \begin{cases} \text{Minimizing } \tau \\ \qquad\qquad \text{s.t.} \begin{cases} A_M{}^T \Delta Q - \tau \leq c_M \\ -A_M{}^T \Delta Q - \tau \leq -c_M \end{cases} \end{cases} \\
&\Longleftrightarrow \quad \text{Minimizing } \mathbf{c}^T y \text{ s.t. } \mathbf{M} y \leq \mathbf{L}
\end{aligned} \tag{11}
$$

where $y = (\Delta Q, \tau)^T, \mathbf{c} = (\underbrace{0, \ldots, 0}_{\text{n times}}, 1)^T \in \mathbb{R}^{n+1}, \mathbf{L} = (c_M, -c_M)^T \in \mathbb{R}^2$ and

$\mathbf{M} = \begin{pmatrix} \mathbf{A_M}^T & -1 \\ -\mathbf{A_M}^T & -1 \end{pmatrix}$ is matrix element of $\mathbb{R}^{2\times(n+1)}$.

When tracking the same risk metric at the sector level, we minimize the $\sum_s \mathcal{R}_M\left(x \mid b, s\right)$. Let $N_{\mathcal{S}}$ be the number of considered sectors. The problem is equivalent to Problem (11) where $y = (\Delta Q, \tau_1, \tau_2, \ldots, \tau_{N_{\mathcal{S}}}), \mathbf{c} = (\underbrace{0, \ldots, 0}_{\text{n times}}, \underbrace{1, \ldots, 1}_{N_{\mathcal{S}} \text{ times}})^T$ are vectors elements of $\mathbb{R}^{n+N_{\mathcal{S}}}$. $\mathbf{L} = (c_M(s_1), \ldots, c_M(s_{N_{\mathcal{S}}}), -c_M(s_1), \ldots, -c_M(s_{N_{\mathcal{S}}}))^T$ is a vector element of $\mathbb{R}^{2N_{\mathcal{S}}}$, and

$$
\mathbf{M} = \begin{pmatrix}
\mathbf{A_M(s_1)}^T & -1 & 0 & \ldots & 0 \\
\mathbf{A_M(s_2)}^T & 0 & -1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{A_M(s_{N_{\mathcal{S}}})}^T & 0 & 0 & \ldots & -1 \\
-\mathbf{A_M(s_1)}^T & -1 & 0 & \ldots & 0 \\
-\mathbf{A_M(s_2)}^T & 0 & -1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
-\mathbf{A_M(s_{N_{\mathcal{S}}})}^T & 0 & 0 & \ldots & -1
\end{pmatrix}
$$

is matrix element of $\mathbb{R}^{2N_{\mathcal{S}}\times(n+N_{\mathcal{S}})}$.

A.3 Tables

Table 10 displays how the MSCI ACWI Index is categorised by stock exchange country and lot size. For instance, in Brazilian stock exchanges, 78 stocks are traded at the minimum lot size of 100. Notably, 93.3% of the stocks have a lot size of either 1 (32.4%) or 100 (60.9%). Stocks traded in Europe have a minimum lot size of 1, while those in the US are 100. Meanwhile, different lot sizes are present in the Hong Kong stock exchange.

Table 11 shows how the Global Aggregate Bond ex-ABS Index is classified by minimum tradable amount and currency. For example, 163 bonds denominated in EUR have a minimum tradable of 0.01. Bonds are mainly traded with a minimum tradable of 1 000 (21.0%), 2 000 (25.7%), 100 000 (25.2%) and 200 000 (11.0%). 2 000 is mainly associated with USD-denominated bonds, while 100 000 is the benchmark for EUR-denominated corporate bonds.

Table 10: MSCI ACWI Index : Breakdown by stock exchange country/Minimum size

	1	5	10	20	50	100	200	250	400	500	1 000	2 000	3 000	4 000	5 000	8 000	10 000	Total	Total (%)
Brazil						78												78	1.0
Canada						306				1								307	3.9
China	32					519												551	7.0
Hong Kong				1	4	39	23	2	13	74	120	83	1	3	1	1	3	368	4.7
Indonesia						51												51	0.6
Japan	51					1 039												1 090	13.9
Malaysia						65												65	0.8
Philippines		1	13			11					2							27	0.3
Singapore						75												75	1.0
Taiwan											179							179	2.3
Thailand						68												68	0.9
United States						2 538												2 538	32.3
Other countries	2 468																	2 468	31.4
Total	2 551	1	13	1	4	4 789	23	2	13	75	301	83	1	3	1	1	3	7 865	
Total (%)	32.4	0	0.2	0	0.1	60.9	0.3	0	0.2	1	3.8	1.1	0	0	0	0	0		100.0

Table 11: Global Aggregate Index: Breakdown by min. tradable/currency

	AUD	CAD	CHF	CNY	EUR	GBP	JPY	KRW	NOK	NZD	Other	SEK	SGD	USD	Total	Total (%)
0.01					163	58					26				247	1.0
0.25					1										1	0.0
1					120									11	131	0.5
8					1										1	0.0
100	26				1						17			283	327	1.3
500					1										1	0.0
1 000	199	1 224	22	1	1 483	182			10	13	137		22	1 976	5 269	21.0
2 000	9	47												6 394	6 450	25.7
3 319.4					1										1	0.0
5 000	25	133	423		2					15	23	10		327	958	3.8
10 000	130	5		311	1	3		127	12	23	58	21		150	841	3.3
25 000														1	1	0.0
50 000			2		31	106	278			6	9	1			433	1.7
100 000	27	12	2	3	5 049	845	202	24		2		13		153	6 332	25.2
125 000					7										7	0.0
150 000		100			11	1								176	288	1.1
200 000	41	25	8		41	29								2 610	2 754	11.0
250 000	11	4			16	6				2			45	303	387	1.5
300 000		6													6	0.0
350 000														1	1	0.0
500 000	5	8									13			1	27	0.1
1 000 000					2	1	32	8	23		67	8		8	149	0.6
2 000 000									5			49			54	0.2
5 000 000					4						19				23	0.1
10 000 000							197								197	0.8
100 000 000							213								213	0.8
1 000 000 000								2			2				4	0.0
5 000 000 000								3							3	0.0
199 999 995 904											2				2	0.0
Total	473	1 564	457	315	6 935	1 231	922	164	50	61	373	102	67	12 394	25 108	
Total (%)	1.9	6.2	1.8	1.3	27.6	4.9	3.7	0.7	0.2	0.2	1.5	0.4	0.3	49.4		100.0

Important Information

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