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About the author

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Marielle heads the fixed-income quant research team in Paris since 2011. Before that she was vice-president of the financial engineering department at Sinopia, an HSBC subsidiary, where she has worked for thirteen years. She started her career in London in 1994 as a research analyst with BARRA and an equity fund manager with Quaestor, a Yasuda subsidiary. She is on the Editorial Board of the Journal of Asset Management.

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Abstract

Mean-variance efficient portfolios are risk optimal only if risk were foreseeable, that is, under the hypothesis that asset price (co)variance is known with certainty. Admitting uncertainty changes the perception. When unforeseen price shocks are deemed possible, risk optimality is no longer synonymous to minimum price variance, but pertains to the diversification in the portfolio as well, for that provides protection against unforeseen shocks.

Generalising Modern Portfolio Theory (Markowitz 1952) in this respect leads to a double risk objective: minimise variance and maximise diversification. The optimum is attained when the portfolio is in parity, meaning that all assets contribute equally to the overall price variance. In that configuration the probability of incurring an unfavourable price shock, foreseen or not, is minimised.

Key words: Modern Portfolio Theory, entropy, diversification, risk parity

Jel code: G11

Investment risk

Back in the Babylonian times it was advocated[1] that households should build up capital proportionally in three domains: a third in the family dwellings, a third in business activity and a third in cash. Investing capital upon the principle of parity was regarded as the best survival strategy in the uncertain world people lived in those days. The story tells in essence what parity investing stands for, it is adopting a holistic approach to asset allocation when prediction power lacks.

The old wisdom that equal weighting is the risk-optimal investment solution under uncertainty has since been formally proven by scientists, e.g. by Bera and Park (2008). In a world where all stochastic variables are uniformly distributed, so that all outcomes are equally likely, exposing uniformly to all is the best defence. In that situation the vulnerability to unfavourable price shocks is minimised. Note that risk refers to unforeseen circumstances in this context, and that optimality is reached when the investment is best protected against that.

Interestingly, equal weighting is not considered risk optimal by modern finance theory standards. An equally-weighted portfolio doesn't possess the return-to-risk potential for it to lie on the efficient frontier (Sharpe 1964). The risk-optimal portfolio is by Markowitz' (1952) definition the fully-invested portfolio with the lowest price variance, and that happens to be a relatively concentrated portfolio, as e.g. Clarke et al. (2011) show. Note that no allusion is made to unforeseen circumstances here and that optimality is reached when the price variance of the portfolio is minimal.

The gap between these two understandings of risk optimality is striking. What exactly makes their divide? We agree with Pola (2016) to say that the answer lies in risk uncertainty, which in investment management terms means that tomorrow's asset price behaviour is not known for certain. Modern Portfolio Theory provides no framework for dealing with uncertain risk parameters; all asset prices are assumed to obey to explicit predefined laws. That assumption makes the divide. Under complete certainty on the risk parameters efficiency is the optimisation objective, whereas in complete uncertainty diversification is sought serving as a protection against unforeseen risk.

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[1] by rabbi Isaac bar Aha and brought to the attention by Shefrin and Statman, see Benartzi and Thaler (2001).

The two prepositions, perfect foresight on risk and none at all, are both highly hypothetical. It is more interesting to study the portfolio optimisation problem under the more plausible condition where risk is partially foreseeable. Asset prices move in line with model parameters … or they don't, giving rise to foreseeable price shocks and unforeseen shocks. An investor seeks protection against those. Investment risk is –reading from the dictionary– the possibility of incurring a financial loss. An investor wants to minimise the probability of incurring a loss due to unfavourable price shocks, which happen to be partly foreseeable, partly not. To do this both objectives must be pursued, we argue, to seek protection as far as unforeseeable risk goes and invest efficiently where risk is foreseeable.

We develop this argument in this article. We relax the assumption of risk certainty in Modern Portfolio Theory, in section 2, and discuss the implications of that for the portfolio optimisation problem. In section 3 we discuss related literature and in section 4 we discuss how the generalised optimisation framework relates to investment practice; section 5 concludes.

Generalising Modern Portfolio Theory

To focus thought, we study, at first instance, the special case of the portfolio optimisation problem where the primary objective is to minimise risk, not to seize performance opportunity. An investor wants to fully invest taking minimal risk so as to gain an overall safe exposure to the capital markets. Without the performance objective, optimality is attained according to Modern Portfolio Theory (MPT) when the price variance of the invested portfolio is minimised, whereby the risk parameters, i.e. the covariance matrix (V) between the asset prices, are assumed certain.

It strikes in this investment setting that the single optimisation objective may not serve the purpose. And indeed it doesn't. The point is that low price variance may contain foreseeable shocks, but not unforeseen shocks. We therefore relax the assumption of risk certainty, by that acknowledging the possibility of unforeseen shocks, and add the objective to seek protection against those. Protection against unforeseeable shocks can only be found in diversification. We borrow this knowledge from the biosciences.[2] In the same

[2] Insight given by Denis Faure, biologist at the CNRS of France.

way that the resilience of an ecosystem augments by its biodiversity, to survive a microbial attack or to regenerate after a forest fire for example, a portfolio is better weathered for price shocks if it contains a wider scale of assets.

We generalise the objective function of the optimisation problem to the one given in equation (1) below. As far as foreseeable risk goes, optimality continues to be defined by the price variance, the first term of the equation, except that it is now specified on a foreseeable covariance matrix V^{θ}, θ indicating the degree of uncertainty. As to unforeseen risk, optimality is defined by the degree of diversification, the second term. There appears to be no standard convention on measuring diversification in the finance profession. Again borrowing from the sciences, we choose in (1) the squared entropy measure (Rao 1982), defined on the portfolio weights (x).

$$\text{min. } x^T V^{\theta} x + \theta \cdot x^T x \tag{1}$$

The degree of uncertainty (θ) sets the relative importance between the two optimisation objectives. It can vary between zero (complete certainty), in which case the function collapses back to the initial MPT setting with a single objective, where $V^0 = V$. Or θ can go to infinite (total uncertainty), at which point the objective reduces to one of diversification only. It can easily be verified that the entropy measure is optimised when all (N) assets are held in equal amounts, $1/N$.

For an intermediate level of uncertainty, $0 < \theta < \infty$, the two objectives are optimised together. This problem has been studied by Maillard et al. (2010) and by Carmichael et al. (2015a, 2015b), who give proof that the optimum is reached when the portfolio is in parity, meaning that all assets contribute equally to the overall price variance. To be precise, the authors specify entropy in a different manner than is done in equation (1), defining a sense of dissimilarity between assets by the Kullback-Leibler divergence measure rather than by the Gini-Simpson measure. For further details we refer to their articles.

The risk-parity portfolio appears to be the most resilient portfolio with respect to price shocks, foreseen or unforeseen. That is valuable insight. It leads to conclude that in an uncertain world the risk-optimal investment strategy is to impose risk parity among the investments that are envisaged. We expand in this article on the implications of this insight for investment management. Risk-parity investing is as it stands presented in the literature as a *smart beta* investment strategy with favourable properties, see Roncalli (2013). We go beyond that; we believe that risk parity is a sensible framework for managing portfolio risk in general.

In fact, when Qian (2006, 2011) introduced risk-parity investing in the mid-2000s, he gave the intuition of minimising financial loss that lies behind it. Though the arguments and empirical evidence he gave were convincing, his argumentation lacked formal proof, as Lee (2011) rightly pointed out. It makes intuitive sense indeed that the optimal trade-off between variance and diversification is reached when all assets contribute equally, that is, equally in terms of marginal price variance ($V \cdot x$) and diversification (x). This can be seen in the example given by Qian of an equity-bond allocation, where volatility is 15% and 5% respectively, and correlation 0.2. The risk-parity allocation, which is at 25 to 75, as $\left(15\%^2 \cdot 0.25 + 0.2 \cdot 15\% \cdot 5\% \cdot 0.75\right) \cdot (0.25) = \left(5\%^2 \cdot 0.75 + 0.2 \cdot 15\% \cdot 5\% \cdot 0.25\right) \cdot (0.75)$, appears better geared for minimising loss than the minimum-variance optimum at 10 to 90.

We now reintroduce the performance objective back into the portfolio optimisation problem. Consider an investor who seeks a safe fully-invested exposure to the capital markets and wants to be rewarded for doing so over the long run. The question is whether the diversification objective is still relevant. Let us see this for the equity-bond allocation question mentioned above. If equities yield 6% per annum over the long run and bonds 0.5%, without the diversification objective the mean-variance optimal allocation with a maximum Sharpe ratio would be 86 to 14. If diversification were added to the optimisation objective, the reader may verify that the optimum would shift to 70 to 30 which is notably different.

We would say that the relevance of diversification fades as soon as tactical performance opportunity is sought. If an investor wants to place an active bet based on market views, the goal of the portfolio optimisation is to implement the views efficiently. In that situation it is coherent to assume complete certainty (θ=0) and maximise the targeted return over targeted risk following Markowitz. Or alternatively to take the Black-Litterman (1992) approach where views and the investor's confidence in these views can be expressed explicitly in the optimisation objective.

To resume, relaxing the certainty assumption in Modern Portfolio Theory leads to a sensible portfolio optimisation framework, which is particularly relevant for passive investment management where the primary purpose is to minimise risk and reap a risk reward over the long run. As can be seen in the equity-bond allocation above, the optimal solution of 70 to 30 seems more reasonable, and is as a matter of fact closer to common investment practice. In diversified funds the ratio tends to be around 60 to 40, see e.g. Lee (2011), and not near the mean-variance optimum.

Related literature

Awareness that Modern Portfolio Theory is somehow falling short, is longstanding and is growing, judging by the number of alternative investment approaches that are being suggested. However, there seems to date no consensus on what exactly is wrong. One of the most repeated critics in the literature is that the Markowitz optimisation problem is vulnerable to estimation error. Bawa et al. (1979) made the point that portfolios which are risk-minimized *ex ante* may not be so *ex post* if the risk estimates turn out to be erroneous. Michaud (1989) went as far as calling portfolio optimizers 'error maximizers'. He showed that small errors in the risk estimates may provoke big changes in the optimised portfolio.

A series of methods have been suggested to reduce estimation error, and in general, doing so improves the portfolio performance. We mention the Bayes-Stein estimation method (Jorion 1986), covariance shrinkage (Ledoit and Wolf 2003) and the more recent stability-adjustment method by Kritzman and Turkington (2016). Scherer (2007) shows that the improvements these methods bring can in essence be attributed to the lesser concentration in the portfolios that are produced. Shrinking the risk parameters, especially the correlation between asset prices, leads *de facto* to more diversified portfolios.

But is the heart of the problem estimation error? Using the term alludes to the idea that the shortcomings of Modern Portfolio Theory lie in its application, not in the theory itself. We don't think so. The postulate that asset price movements are stationary processes which obey to predefined statistical laws, qualifying deviation from that as error, is a misconception. Taking the view that price movements cannot be fully anticipated *ex ante* and integrating that view into the definition of investment risk, changes the concept. The heart of the problem is the absence of uncertainty, and that is not estimation error.

By admitting uncertainty unforeseen risk becomes part of the investment problem, and inherent to that, diversification enters the optimisation objective. We underline the importance of defining diversification on intrinsic characteristics of assets, such as weights or industry categories, expressly not on risk parameters. It is easy to see why. As soon as prices deviate from the predefined parameters, the diversification becomes suboptimal by its own definition and the protection less effective with it. The diversification measure proposed by Meucci (2009) based on eigenvalues would be inappropriate for this reason, as the diversification ratio defined by Choueifaty and Coignard (2008).

A number of optimisation methods have been proposed in the literature that augment the portfolio diversification, such as resampling (Michaud 1998) and robust optimisation (Tütüncü and Koenig 2004). Resampling means taking the average over a set of portfolios that are optimised on different risk estimates. In the same spirit, in a robust optimisation a portfolio is optimised not with respect to one risk model that is plausible but to a set of models that are equally plausible. We note that in both methods the assumption of risk certainty is relaxed in a sense, albeit in an *ad hoc* way.

Generalised Modern Portfolio Theory in practice

As was seen in the example given in section 2, the solution to the generalised optimisation problem appears more in line with investment practice than the mean-variance optimum. Risk uncertainty is a real issue for the equity-bond allocation problem. Correlation between equities and bonds is unstable, and this is so as a result of conflicting market forces; on the one hand correlation is positive as the valuation of both asset classes depends on the state of the economy, on the other hand a direct substitution effect drives towards negative correlation.

Fortunately the correlation parameter doesn’t play a role in the generalised optimisation problem when it concerns two assets; Maillard et al. (2010) make this evident. The instability doesn’t affect the optimisation outcome. The parameters that are determinant are the volatility- and yield levels. These are stable and easy to anticipate over long horizons, which makes the optimisation robust and reliable. In addition, the risk-parity solution is by its nature less sensitive to the parameter inputs than the mean-variance optimum. It suffices to get the order of magnitudes right to obtain sensible asset allocations.

Let us investigate a broader investment setting where an allocation is to be decided over a multiple of assets, for example, let us look at a risk-parity strategy applied onto equities. It is, as we argue in section 2, the risk-optimal procedure to gain a global equity market exposure. Clarke et al. (2013) derive the near-closed form solution to this problem in case the Capital Asset Pricing Model (Sharpe 1964) is adopted to span the covariance matrix V^{θ}. Adopting this model expresses the view that the market factor is certain and nothing else. They demonstrate that in that case the portfolio weights are inversely proportional to the market betas, which is an intuitive result.

Other factors than the market may be considered for inclusion in the risk model. Whether to add a factor depends, we argue, on the question whether it is more damaging for the portfolio performance to ignore the factor so that risk efficiency is partly foregone, or to adopt it, in which case the protection is partly sacrificed. That question needs further thought. In any event it shouldn't be the same risk model that is adopted with or without the certainty assumption, i.e. $V^{\theta} \neq V$ unless $\theta=0$. It would be incoherent to carry out a generalised optimisation (thus impose risk parity) based on the same risk model as one would use in a mean-variance optimisation.

For the generalised optimisation problem, apart from the risk model the dissimilarity between the assets must be specified. Opting for a default setting where all assets are equally dissimilar leads to the default risk-parity solution. But it is possible to go beyond that, to consider assets belonging to the same industry group more similar than others for example. That again needs further thought. The underlying question is whether it is more beneficial to specify a risk factor based on similar price behaviour or specify a sense of similarity on directly-observable characteristics.

In fact, the argument we make in this article that the optimisation objective adapts to the degree of uncertainty, coincides with investment practice. Consider the top-down approach, generally pursued when making strategic allocation decisions for a globally invested fund. The allocation over countries, asset classes and, within those, over individual assets is decided layer by layer. On the aggregate level, where risk parameters are least certain, the risk-parity rule tends to be applied, while on the same continuum, on assets mean-variance optimisation tends to be applied. In the light of generalised optimisation proceeding this way is rational and coherent.

Conclusion

In this article we place the portfolio optimisation problem in a wider context than its initial specification. Rather than staying with a model where the risks in a given market are predefined by statistical laws, the possibility of escaping from those laws is considered part of the problem definition. Opening up and relaxing the definition of investment risk leads to a portfolio optimisation framework that is sensible and coincides with investment practice.

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